August 2014 Physics & Astronomy Qualifying Exam
for Advancement to Candidacy
Day 1: August 28, 2014

Do not write your name on the exam. Instead, write your student number on each exam booklet. This will allow us to grade the exams anonymously. We'll match your name with your student number after we finish grading. If you use extra exam booklets, write your student number on the extra exam books as well. Write all answers in the blank exam booklet(s), not on this printout!

Today's portion of the exam has 8 questions. Answer *any five* of the eight. Do not submit answers to more than 5 questions—if you do, only the first 5 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 4 hours to complete 5 questions.

You are allowed to use one $8.5'' \times 11''$ formula sheet (both sides), and a handheld, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	-273°C
air pressure at sea level	1 atm	10^5 N/m^2
atomic mass unit	1 amu	1.66×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
electron volt	1 eV	1.6×10^{-19} J
Laplacian in spherical coordinates	$\nabla^2 f =$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
Larmor formula		$P_{rad} = \mu_0(\ddot{m})^2/6\pi c^3$
mass of an electron	m_e	0.511 MeV/c^2
mass of a proton	m_p	938 MeV/c^2
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
nuclear magneton	μ_N	5×10^{-27} J/T
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the Earth	R_{earth}	6.4×10^6 m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Stirling's approximation	$N!$	$e^{-N}N^N\sqrt{2\pi N}$

1. Consider a spherical unicellular organism. Oxygen is transported through the cell by diffusion, as governed by the diffusion equation:

$$D\nabla^2 N(\vec{x}, t) = \frac{\partial N}{\partial t}$$

where $N(\vec{x}, t)$ is the concentration of oxygen (molecules/m^3) at any point in space or time and D is a constant.

Assume that oxygen is consumed at a constant uniform rate α [units: molecules/(m$^3\cdot$s)] throughout all parts of the cell's interior. Suppose that the cell is bathed in a fluid with constant oxygen concentration N_0.

Calculate the largest possible size of the cell if the steady state oxygen concentration in all parts of the cell must be greater than some level N_{min}. You may assume that oxygen transport across the cell's membrane occurs through diffusion at the same speed as diffusion through the interior of the cell.

2. Order of magnitude estimation: Could a snowflake cooled to 10 microkelvins be lifted with an ordinary permanent magnet, 10 cm across, with a strength of 0.2 T measured at a distance of 10 cm from either pole, acting on the induced nuclear polarization? The proton's magnetic moment is $2.8\mu_N$. Oxygen is a spin-0 nucleus. Ignore any contributions from electrons. Explain fully your reasoning.

3. A spin is initially prepared in the $|s = 1/2, s_z = +1/2\rangle$ state. A periodically reversing magnetic field with period T is applied along the x direction. For each period, $B = B_0$ when $T/2 > t > 0$ and $B = -B_0$ when $T > t > T/2$. Find $\langle S_y \rangle$ and $\langle S_x \rangle$ at $t = (n + 1/2)T$ and at $t = nT$, where n is an integer.

4. A thin door of height H and width D is suspended on its hinges whose axis $\hat{z}'$ is inclined to the vertical $\hat{z}$ by an angle θ. What is the period of (small) oscillations of this door as it flaps back and forth around axis $\hat{z}'$?

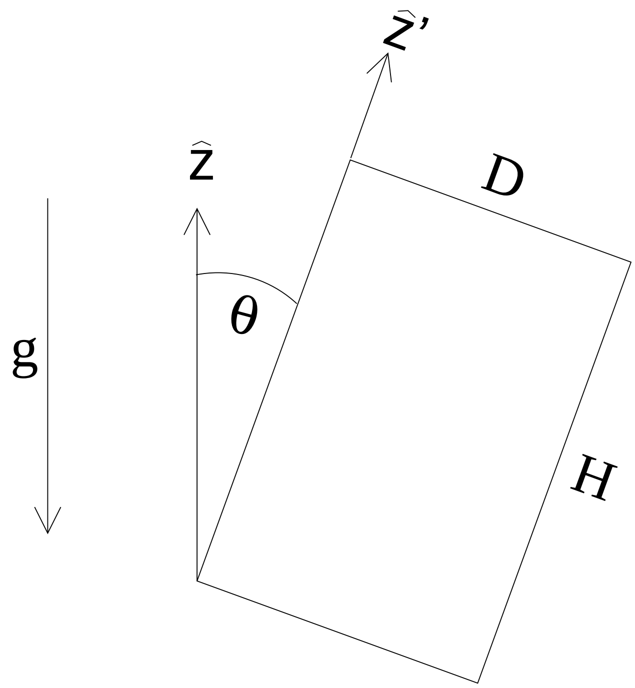
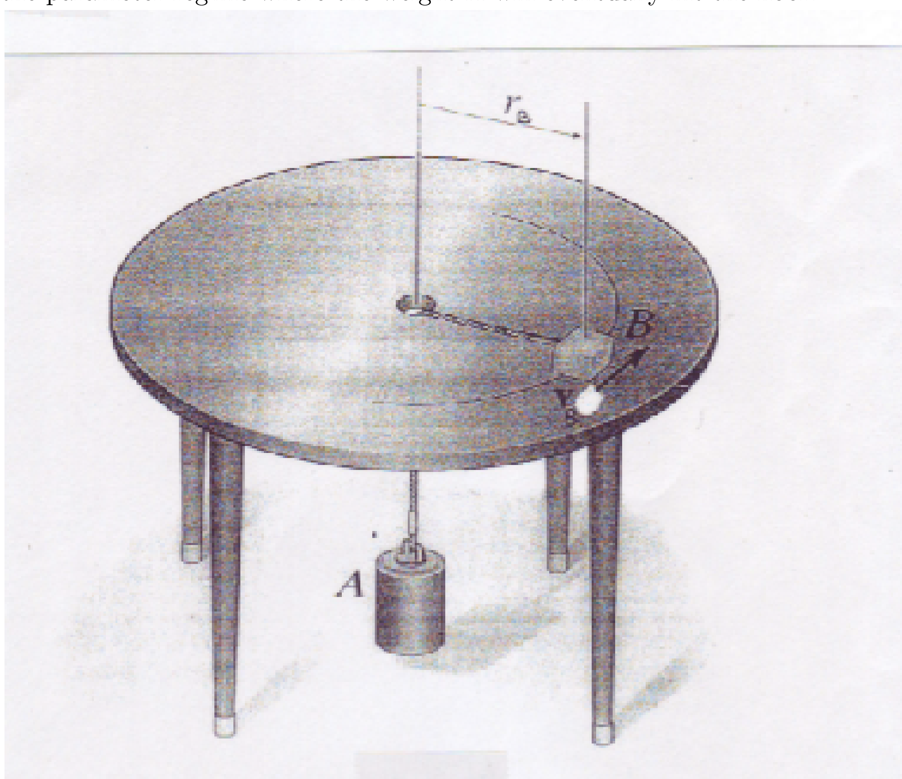


Figure 1: Swinging door tilted at an angle.

5. A weight B with mass m_B lies on a frictionless table as depicted in the diagram below. It is connected to a cord which is threaded through a hole in the centre of the table. The weight B is initially set in circular motion with an initial velocity v_B and at radius r_B. The other end of the cord is attached to a weight A of mass m_A which is initially at rest a height h above the floor. Then the weight is released from rest, and it will exert a downward force on the cord. Describe the subsequent motion of the weight A. Find an equation for the parameter regime where the weight A will eventually hit the floor.



6. As a first approximation, the neutron-proton interaction can be described by the attractive square-well potential

$$V(r) = -V_0\theta(a-r)$$

where $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ if $x < 0$. This potential has two parameters, the depth of the well V_0 and the range of the interaction a. The range is of order the pion Compton wavelength $a \approx \lambda_\pi = \hbar/(m_\pi c)$ with $m_\pi c^2 =$ 139 MeV and $\hbar c = 197$ MeV·fm . Estimate V_0 using the fact that the neutron and proton have only one bound state, the deuteron, with binding energy of $-2.22 MeV$. Use a proton mass equal to a neutron mass, $m_N = m_P = 940$ MeV.

7. A heat pipe is a tube containing a small amount of a liquid and the balance of the volume filled with the liquid's vapor, and is used to transfer heat efficiently in confined spaces. For example, heat pipes are often used to cool the processor in laptops. In this problem, a water-based heat pipe is 5 mm inner diameter and 30 cm long, and held in a vertical orientation so most of the liquid is at the bottom of the tube. Heat can be transferred very rapidly along the heat pipe if the liquid end is heated and the other end cooled, as liquid from the hot end evaporates, then recondenses when it reaches the other end. Provide an estimate of the maximum rate of heat transfer by such a heat pipe, including carefully spelling out assumptions you are making. Try to provide numerical estimates for as many parameters as possible that go into your calculations.

8. The outer 100 km of the earth is nearly solid rock (the lithosphere). The inner boundary of the lithosphere is defined by the depth where the temperature becomes so hot (around 2000°C) that rock begins to flow. Within this inner boundary, liquid rock can flow, transfering heat by convection so temperature gradients are much smaller. (The core temperature reaches 5000-7000K.) An order of magnitude estimate for thermal conductivity in rock is 1 W/m/K. The primary source of heat in the interior of the earth is believed to be radioactive decay of various elements, especially ^{232}Th and ^{238}U. Estimate:

A. by how much the outer surface temperature of the earth is raised due to this inner heat

B. roughly how much heat is generated per cubic meter within the earth.

August 2014 Physics Qualifying Exam
for Advancement to Candidacy
Day 2: August 29, 2014

If you are in the Ph.D. in astronomy program, don't write this exam! Ask a proctor for the astronomy version of today's exam!

Do not write your name on the exam. Instead, write your student number on each exam booklet. This will allow us to grade the exams anonymously. We'll match your name with your student number after we finish grading. If you use extra exam booklets, write your student number on the extra exam books as well. Write all answers in the blank exam booklet(s), not on this printout!

Today's portion of the exam has 8 questions. Answer *any five* of the eight. Do not submit answers to more than 5 questions—if you do, only the first 5 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 4 hours to complete 5 questions.

You are allowed to use one $8.5'' \times 11''$ formula sheet (both sides), and a handheld, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	-273°C
atomic mass unit	1 amu	1.66×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
electron volt	1 eV	1.6×10^{-19} J
Laplacian in spherical coordinates	$\nabla^2 f =$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
Larmor formula		$P_{rad} = \mu_0(\ddot{\vec{m}})^2/6\pi c^3$
mass of an electron	m_e	0.511 MeV/c^2
mass of a proton	m_p	938 MeV/c^2
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
nuclear magneton	μ_N	5×10^{-27} J/T
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the Earth	R_{earth}	6.4×10^6 m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}

9. If you rub a balloon against your hair and then bring it close to a stream of water flowing from a faucet, you will observe the water to bend towards the balloon (as in the figure).

Figure 2: A stream of water bending towards a balloon.

(a) Why does this occur? Describe how the electrical charges are distributed on the balloon and in the water in order to produce the attractive force observed.

(b) In this situation, why does the water bend toward and not away from the balloon? Under what condition(s) could the water be made to bend away from the balloon?

10. A electric eel of length 2 m generates a voltage difference between its head and tail using 5000 electrocyte cells. These cells are connected in series, and each produces a potential difference of 140 mV during a shock, as well as having an internal resistance. When measured in air (out of water!), the open circuit potential difference between the head and tail is measured to be 600 V. Note that some "leakage current" flows back through the skin of the eel. If the tail is then "shorted" to the head through an ammeter, the measured current is 1 amp.

A. Draw a Thevenin equivalent circuit for the eel, giving numerical values for all of its components. (Hint: A Thevenin equivalent circuit consists only of resistors and ideal voltage sources.)

B. The eel is released into freshwater with a resistivity of 200 $\Omega \cdot m$. Do an order of magnitude estimate of the electrical resistance to the current flowing through the water, and use this to calculate the current pushed by the eel through the water.

C. An electric ray lives in saltwater, which has a resistivity of 0.23 $\Omega \cdot m$. From the standpoint of delivering the maximum power to the water, explain why the electric ray will produce a lower voltage (around 50 V) but higher current than the eel. (Note: the ray has the same number of electrocytes as an eel, but places some of them in parallel rather than all in a series.)

11. Rotating liquid mirror

A. A rotating container of radius R holds reflective liquid mercury ($\rho = 13.6\ \mathrm{g/cm^3}$) at its bottom. In its steady state, the container and the mercury all rotate at the same angular velocity ω. and the surface of the mercury acquires a concave curvature. By considering how the centripetal acceleration of a small volume of fluid at the surface is provided by the radial component, calculate the height of the fluid's surface as a function of the distance from the axis of rotation.

B. This curved surface acts as a mirror. If the cylinder has a diameter of 2 m and rotates at 20 rotations per minute, determine the focal length of this mirror (which has its axis in the vertical direction).

12. Find the distribution of the total momentum $\vec{P}$ of N identical free particles at temperature T. Calculate the most probable speed of the centre of mass of the collection. Hint: The velocity distribution for each particle follows the Maxwell distribution.

13. On July 16, 1945, the United States detonated the first atomic bomb in a test code-named Trinity. British physicist G.I. Taylor estimated the yield of the nuclear explosion from a photograph of the fireball using simple dimensional analysis. He reasoned that the radius R of the blast should initially depend only on the energy E of the explosion, the time t after the detonation, the density $\rho = 10^{-3}$ g/cm^3 of the air, and a dimensionless constant, S, related to the ratio of specific heats of air.

Find an expression for $R(t) = Sf(E, \rho, t)$. Given that S is of order unity, find the approximate energy, E, produced in the explosion. Express your result in tons of TNT where 1 ton TNT $= 4.2 \times 10^9$J.

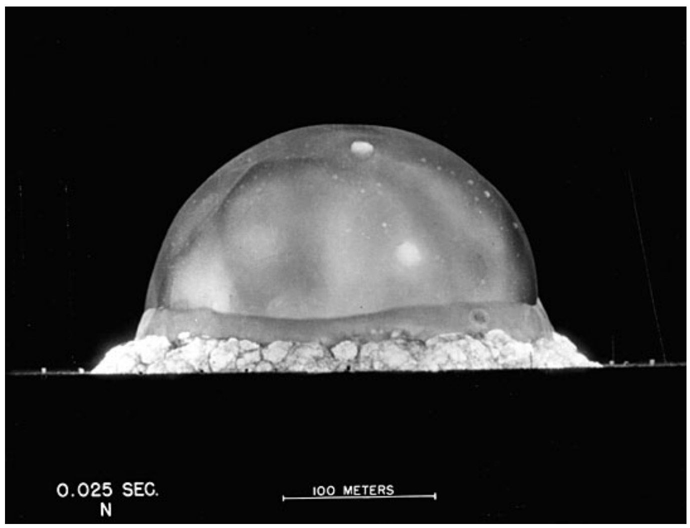


Figure 3: The 1945 TRINITY nuclear explosion fireball . Picture taken 25 ms after detonation with a Rapatronic camera.

14. Suppose you have a cloud of bosonic ultra-cold atoms confined in a 3-dimensional harmonic potential, and at time $t = 0$ the confinement is switched off. The atoms will certainly fall, due to gravity, and in that falling inertial frame, the cloud will also expand.
a) At a temperature well above the critical temperature for Bose-Einstein Condensation, what will be the shape of the falling, expanding cloud once its size has grown much larger than its initial size in the trap? How does this size depend on the mass of the atoms, the initial temperature T and the time of the expansion t?

b) At a temperature below the critical temperature, Bose-Einstein condensation will occur. Even at a vanishingly small temperature, the condensed atomic cloud will also expand when the confinement is switched off. Briefly explain why. Assuming the initial confining potential is harmonic, what will be the shape of the falling, expanding condensate once its size has grown much larger than its initial size in the trap? How does this size depend on the mass of the atoms, the initial cloud size and the time of the expansion?

c) Suppose now that an atomic BEC is split into two clouds separated by a distance d, which is much larger than the size of either cloud. At time $t = 0$, the harmonic confinement for each cloud is switched off and the atoms fall due to gravity and also expand. In the volume where the clouds overlap, matter-wave interference is observed. Briefly explain why and compute the shape of the interference fringes. To make things easier, you can model the two clouds as point sources of atoms and work in the inertial (falling) reference frame. To deduce the shape of the fringes, think just about the phase of the waves and don't worry about the overall amplitude of the wave function.

15. A rope wraps around two fixed cylinders as shown in the figure. The cylinders are made from two different materials and their coefficients of static friction are μ_1 and μ_2. Cylinder one and two have a radius of 2 cm and 7 cm, respectively. A mass of 54 kg hangs from the end of the rope attached to cylinder 2 and a mass of 1 kg is attached to the other end. Find the conditions on the coefficients of friction μ_1 and μ_2 such that the masses do not slip.

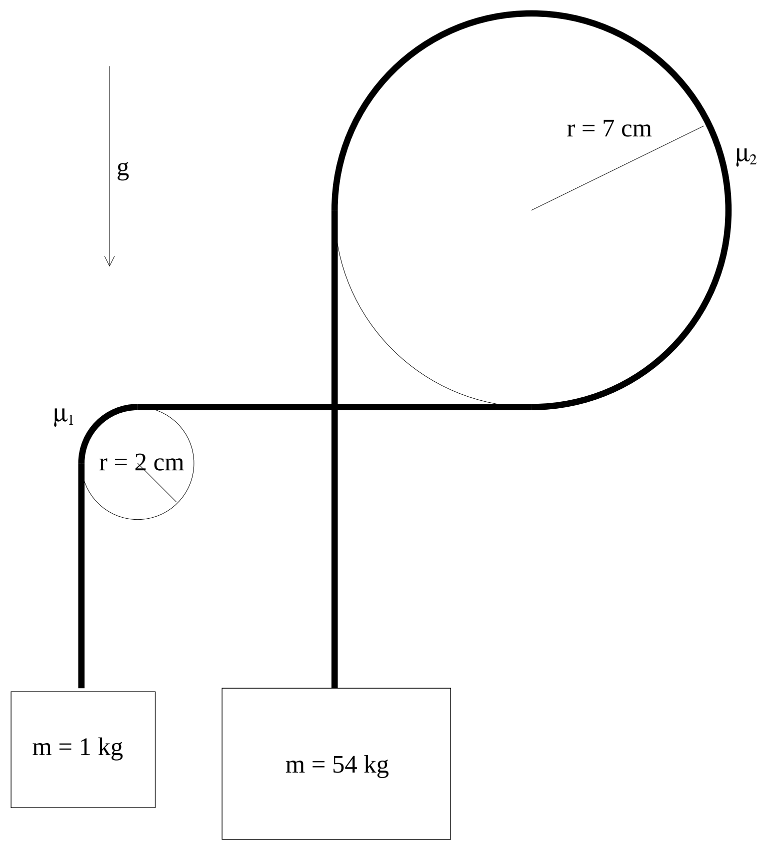


Figure 4: Ropes and pulleys.

16. Electron and τ-neutrino mixing is described by the Hamiltonian

$$H_{\text{mixing}} = -\omega\left[\sin(2\theta)\sigma_3 + \cos(2\theta)\sigma_2\right] + \frac{GN}{\sqrt{2}}\sigma_3$$

where σ_i are Pauli matrices, $\omega = (m_\tau^2 - m_e^2)/2E$ with E the relativistic neutrino energy, θ is the vacuum mixing angle, G is the Fermi decay constant, N is the density of electrons in matter and the electron neutrino is the state $\begin{bmatrix}1\\0\end{bmatrix}$ and the τ neutrino is the state $\begin{bmatrix}0\\1\end{bmatrix}$.

If the electron density in matter N is constant, at what distance L = ct would all electron neutrinos entering matter be converted to tau neutrinos?

Hint: the Pauli spin matrices are:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$