August 2012 Physics & Astronomy Qualifying Exam
for Advancement to Candidacy
Day 1: August 30, 2012

Do not write your name on the exam. Instead, sign up on the sign-up sheet, and note the serial number next to your name. Write that number on your exam in place of your name. This will allow us to grade the exams anonymously. We'll match your exam with your name using the sign-up sheet after we finish grading. If you use extra exam booklets, write your serial number on the extra exam books as well.

Today's portion of the exam has 8 questions. Answer *any five* of the eight. Do not submit answers to more than 5 questions—if you do, only the first 5 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 4 hours to complete 5 questions.

You are allowed to use one $8'' \times 11''$ formula sheet (both sides), and a hand-held, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	-273°C
air pressure at sea level	1 atm	10^5 N/m^2
atomic mass unit	1 amu	1.66×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
distance from earth to moon	d_{moon}	3.8×10^8 m
distance from earth to sun	1 AU	1.5×10^{11} m
electron volt	1 eV	1.6×10^{-19} J
Larmor formula		$P_{rad} = \mu_0(\ddot{m})^2/6\pi c^3$
mass of an electron	m_e	0.511 MeV/c^2
mass of the moon	M_{moon}	7×10^{22} kg
mass of the sun	M_{sun}	2×10^{30} kg
mass of a proton	m_p	938 MeV/c^2
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the Sun	R_{sun}	7×10^8 m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}

1. Consider a particle of mass m moving in a 1D attractive delta potential $V(x) = V_0\ \delta(x)$ at the center of a finite box potential $(-L < x < L)$, with $V(x) = \infty$ for $|x| > L$.

A) When L is infinite, calculate the energy and eigenfunction of the lowest energy state.

B) Now consider a finite box. Find out the condition under which there will be a state with a negative energy. Can you give an interpretation of why only at a particular size, L_c, a bound state starts to appear in the spectrum?

2. The dark matter halo of our galaxy can be modelled as a non-relativistic thermal gas of massive particles with mass M_X at temperature T. The Earth moves through this gas with a velocity of $v_0 = 230$ km/s.

A. Determine the distribution of speeds $F(s)$ of the dark matter particles relative to the Earth. (You don't need to normalize the distribution.)

B. A dark matter particle collides elastically with a nucleus at rest of mass M_N, and the nucleus recoils with a (non-relativistic) kinetic energy of E_R. Calculate the minimum speed of the dark matter particle needed to produce a recoil of this energy.

C. In an inelastic collision, the dark matter particle absorbs some of the collision energy to reach an excited state with mass $M_X + \Delta$, while the nucleus recoils with kinetic energy E_R. Calculate the the minimum needed speed of the dark matter particle for this to happen.

3. Iron atoms (atomic mass 56) contain two free electron spins that can align with an external magnetic field. An iron wire 3 cm long and 1 mm in diameter is suspended vertically and is free to rotate about its axis. A strong magnetic field parallel to the wire's axis is applied. How large is the resulting change in its angular velocity? The density of iron is 8 g/cm^3.

4. A neutron star can be modelled as a spinning, radiating magnetic dipole. Given the value of the spin period of a neutron star and the change of the spin period with time, obtain an order of magnitude estimate of the strength of the magnetic field at the surface of the star. A neutron star has a radius of about 10 km and a mass of about 3×10^{30} kg. Suppose the period is $P = 0.01$ s and the spindown rate is $dP/dt = 10^{-12}$.

5.

A. Show that if a black body radiation field exists inside an expanding cavity, at some time later it will still be black body but at a lower temperature. Show explicitly that for an expanding box of side length L that $dL/L = -dT/T$.

B. If the wavelength of each photon is Doppler-shifted because of motion of an observer with respect to the frame of reference defined by the 3 degree K cosmic microwave background, show that the measured temperature T_{rad} of the cosmic microwave background should vary sinusoidally with angle, having a peak fractional amplitude v/c.

6. Normal blood pressure for a resting person is approximately 100 Torr (1 Torr=133 Pa). The human heart has four chambers with a volume of 70 milliliters each. Using these numbers and common knowledge about the heart estimate the following:

A. The minimum pump power needed by the heart.

B. How many kCal of energy are needed each day to power the heart (1 kCal = 4184 J)

C. Roughly how many g's of upwards acceleration would cause a person in an upright position to black out?

7. A circular hoop of radius R is charged with a linear charge density of λ. The hoop is rotated with angular velocity ω about the z-axis, which passes through a diameter of the hoop.

A. Calculate the quadrupole moments Q_{XX}, Q_{YY}, Q_{ZZ} of the distribution at the point when the hoop lies in the XZ plane.

B. How does the amount of radiated power vary with λ?

C. If λ is kept fixed but the radius is doubled, by what factor is the radiated power changed?

D. If the hoop is instead rotated about its center through an axis perpendicular to the plane of the hoop, by what factor does the radiated power change compared to when it is rotated through an axis crossing the center in the plane of the hoop?

8. Combine the fundamental constants G, $\hbar$ and c to give quantities with the dimensions of mass, length and time. What are the values of these “natural” or “Planck” fundamental constants in MKS units?

August 2012 Physics & Astronomy Qualifying Exam
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Day 2: August 31, 2012

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mass of the Sun	M_{sun}	2×10^{30} kg
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permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
Planck's constant, reduced	$\hbar$	1.1×10^{-34} J·s
radius of the Sun	R_{sun}	7×10^8 m
specific heat of water	C	4186 J/kg/°C
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}

9. A mass m is attached by a massless rod of length ℓ to a pivot, which allows the mass to swing freely in a vertical plane under the influence of gravity. Let θ be the angle between the rod and the vertical.

A. In the small-angle approximation, find the energy levels of the system.

B. Find the lowest-order correction to the ground state energy resulting from the inaccuracy of the small-angle approximation. You may leave your answer in the form of a definite integral.

10. An electron in a one-dimensional infinite square potential well of width $a = 100$ nm has the following initial wavefunction:

$$\psi(x,0) = Ax(a-x)$$

A. Find A.

B. How does this wavefunction evolve in time? Find $\psi(x,t)$ (it is OK to use a series expansion and solve for the coefficients in the form of definite integrals that you don't evaluate.). Will the wavefunction ever return to the above initial state? If so, how long will it take?

C. What is the probability that a measure of the energy of the electron will result in a value of exactly 1 meV? The mass of an electron is $9 \times 10^{-31} kg$ and $h = 6.6 \times 10^{-34} J - s$.

11.

The astronomical unit is currently defined as the distance at which a massless particle would orbit the Sun with a period of 1 year.

(a) Prove the following relation for a particle orbiting the Sun:

$$\frac{2\pi}{P} = \sqrt{\frac{GM}{a^3}}$$

Here P is the orbital period, a is the semimajor axis (or radius for a circular orbit) of the test particle, and GM is the product of Newton's gravitational constant G and the Sun's mass M.

Note that because G is very difficult to measure (only known to about 5 significant figures in laboratory experiments) it is the product GM (which always appears together) that is measured extremely well (to about 12 significant figures). Given that definition, to low precision the AU is roughly 1.498×10^{11} meters, and the solar mass about 1.99×10^{30} kg.

(b) A proposal before the Standards Commission of the International Astronomical Union this summer proposes defining the astronomical unit as a fixed number of SI meters. The reason is that the quantity GM (on which the AU is based) is not constant because the Sun is losing mass mostly via the mass-energy ($E = mc^2$) in the photons departing it.

Consider the Earth on a circular orbit around the Sun. You may assume without proof that the orbit remains circular and that the angular momentum per unit mass of the Earth remains constant during the Sun's mass loss. Estimate the effect on the Earth's position over a 10 year interval if Sun is losing mass via spherically symmetrical radiation.

If one can measure the Earth's orbital radius to 1 meter accuracy, would a fractional mass loss rate estimated at 10^{-13} parts per year be detectable? Show your work and state your assumptions.

12. *Quantum mechanical harmonic oscillator*

Consider a quantum mechanical particle of mass m in a potential $V(\mathbf{r}) = \kappa \mathbf{r}^2/2$ of a harmonic oscillator in 3D with a spring constant $\kappa > 0$. Find a solution for the stationary Schrödinger equation using the following ansatz:

$$\phi(x, y, z) = bx \exp\{-a(x^2 + y^2 + z^2)\} \tag{1}$$

where a and b are constant. Calculate the energy E of this state and the coefficient a as functions of $\kappa, \hbar$ and m.

Are we dealing with an s, p, d or f orbital?

13. *“Triangle” process*

Consider the following thermodynamic cycle in a monatomic ideal gas of N particles:

A. Start at initial pressure $p_A = p_0$ and initial volume $V_A = V_0$ (state A).

B. Now, halve the pressure in a cooling process at constant volume, e.g. $p_B = p_0/2$ (state B).

C. Then, halve the volume in an isothermal compression: $V_C = V_0/2$ (state C).

D. Finally, perform an isobaric (constant pressure) expansion back to state A.

a) Sketch the cycle A→B→C→A in a p-V diagram and give p, V, T, and the internal energy U for all states A, B, C.
b) Calculate for each transition (A→B, B→C, and C→A) the work W, the change in internal energy ΔU, and the heat Q. Explicitly state whether the gas is performing work on the environment or vice versa, as well as whether it is donating or accepting heat.
c) Calculate the efficiency η and compare to the Carnot efficiency.

14. *Relativistic point mass*

A particle with rest mass m_0 moves in 1D under the influence of a force (specified in the lab frame) decreasing exponentially with time:

$$F(t) = F_0 \exp(-\kappa t), \tag{2}$$

where F_0 =const and κ =const.

a) Calculate $v(t) = \dot{x}(t)$ if starting from rest: $v(0) = 0$.

b) Give the energy $E(t)$ and momentum $p(t)$ of this particle in the lab frame as a function of time.

c) From the constants F_0, m_0, κ, and c one can calculate a dimensionless quantity χ:

$$\chi = \frac{F_0}{\kappa m_0 c}.$$

For which limiting case of χ will the motion be (i) non-relativistic and (ii) ultra-relativistic?

15. The inside of your eyeball contains electromagnetic radiation from two sources: visible photons that have entered the pupil, and 310 K black body radiation. (310 K = normal body temperature). Do an order of magnitude estimate of the ratio of the total energy in the eye due to photons entering through the pupil to the energy of photons from the black body radiation, if the eye is open in a well-lit room.

16. Neutrons are spin-1/2 particles which have a magnetic moment given by $(ge/2m_p)\vec{S}$, where $\vec{S}$ is the spin angular momentum of the neutron and $g \approx -3.8$.

A. A strong magnetic field is applied in the x direction to a group of neutrons that have been cooled to a temperature of $T = 1$ mK. What field strength should be applied so that the neutrons are 99% polarized?

B. With the neutrons polarized in the x direction as in part A, the magnetic field is instantaneously replaced by a 1 Tesla field orientated in the z direction. The polarization of the neutrons begins to precess around the B field. What is the frequency of this precession?