September 2010 Physics & Astronomy Qualifying Exam
for Advancement to Candidacy
Day 1: September 2, 2010

Do not write your name on the exam. Instead, sign up on the sign-up sheet, and note the serial number next to your name. Write that number on your exam in place of your name. This will allow us to grade the exams anonymously. We'll match your exam with your name using the sign-up sheet after we finish grading. If you use extra exam booklets, write your serial number on the extra exam books as well.

Today's portion of the exam has 8 questions. Answer *any five* of the eight. Do not submit answers to more than 5 questions—if you do, only the first 5 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 4 hours to complete 5 questions.

You are allowed to use one $8'' \times 11''$ formula sheet (both sides), and a hand-held, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

| | | |
|---|---|---|
| absolute zero | 0 K | -273°C |
| air pressure at sea level | 1 atm | $10^5$ N/m$^2$ |
| atomic mass unit | 1 amu | $1.66 \times 10^{-27}$ kg |
| Avogadro's constant | $N_A$ | $6.02 \times 10^{23}$ |
| Boltzmann's constant | $k_B$ | $1.38 \times 10^{-23}$ J/K |
| charge of an electron | $e$ | $1.6 \times 10^{-19}$ C |
| electron volt | 1 eV | $1.6 \times 10^{-19}$ J |
| Larmor formula | | $P_{rad} = (\ddot{\vec{m}})^2/6\pi\epsilon_0 c^3$ |
| mass of an electron | $m_e$ | 0.511 MeV/c$^2$ |
| mass of a proton | $m_p$ | 938 MeV/c$^2$ |
| mass of the Sun | $M_{sun}$ | $2 \times 10^{30}$ kg |
| Newton's gravitational constant | $G$ | $6.7 \times 10^{-11}$ N m$^2$ kg$^{-2}$ |
| permittivity of free space | $\epsilon_0$ | $8.9 \times 10^{-12}$ C$^2$ N$^{-1}$/m$^2$ |
| permeability of free space | $\mu_0$ | $4\pi \times 10^{-7}$ N/A$^2$ |
| Planck's constant | $h$ | $6.6 \times 10^{-34}$ J·s |
| Planck's constant, reduced | $\hbar$ | $1.1 \times 10^{-34}$ J·s |
| radius of the Sun | $R_{sun}$ | $7 \times 10^8$ m |
| speed of light | $c$ | $3.0 \times 10^8$ m/s |
| Stefan-Boltzmann constant | $\sigma$ | $5.67 \times 10^{-8}$ W m$^{-2}$ K$^{-4}$ |

1. Order of magnitude estimation

It is estimated that 3,000 meteors with enough mass to destroy an airplane hit the Earth every day. There are 20 million commerical airline flights a year. Estimate the chance that an airplane in flight will be destroyed by a meteor sometime in the next 20 years. An order of magnitude calculation will suffice.

2. Consider two operators, $A$ and $B$ which satisfy:

$$[A, B] = B \quad ; \quad B^\dagger B = 1 - A^2$$

A. Determine the hermiticity properties of $A$ and $B$.

B. Using the fact that $|a = 0>$ is an eigenstate of $A$, construct the other eigenstates of $A$.

C. Suppose the eigenstates of $A$ form a complete set. Determine if eigenstates of $B$ can be constructed, and if so, determine the spectrum of the eigenstates of $B$.

3. A soccer ball with radius $R = 11$ cm is inflated to a gauge pressure of $9 \times 10^4$ Pa. The ball is dropped onto and bounces elastically off of a hard smooth floor. Find approximate expressions for the surface area of the ball in contact with the floor, the amount of time the ball is in contact with the floor, and the peak force exerted on the floor, if the mass of the ball is 0.42 kg and it is dropped from a height of 0.1 m onto the floor.

4. A pulsar with mass $M$, radius $R$, and rotational period $P$ can be modelled as a precessing magnetic dipole tilted at angle $\alpha$ with respect to the rotation axis. Suppose that the $B$ field strength at the surface is $B_0$. The precessing dipole radiates energy, causing the star's rotation rate to slow down at rate $dP/dt$.

A. Near the pulsar, $B(r) = B_0(r/R)^n$. What is $n$ for a dipole-like field?

B. Assume that this relation is valid out to the "light cylinder", that value of $r$ where an object circling the neutron star with period $P$ moves at the speed of light, $c$. Evaluate $B$ at the light cylinder, roughly at the equator, in terms of $B_0$, $R$, $P$, and $c$.

C. Now assume that, exactly at the light cylinder, $B$ makes a transition from a static dipole inside the light cylinder to a propagating electromagnetic field outside. What is the total energy lost per unit time ($dE/dt$)?

D. Now equate your formula for $dE/dt$ to the kinetic energy lost by the neutron star as its rotation slows down, and derive an equation for $B_0$ in terms of $M$, $R$, $c$, $P$, and $dP/dt$.

5. Quarks inside a nucleon can be modelled by a "bag model" in which they are considered to be trapped inside a spherical shell with radius $R$ by a spherically symmetric potential of the form

$$V(r) = \begin{cases} 0 & \text{if } r < R \\ \infty & \text{if } r \geq R \end{cases}$$

A. What is the energy of the lowest energy bound state of a quark of mass $M$ with $\ell = m = 0$?

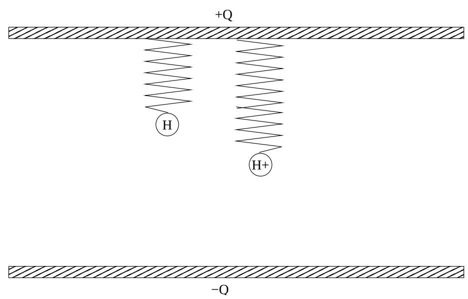
B. How much pressure does the quark exert on the surface of the sphere in this case?

C. Now consider instead a spherical potential of the form $V(r) = -\gamma\delta(r-a)$, where $\gamma$ and $a$ are positive constants. Determine the smallest value of $\gamma$ for which there is precisely one bound state. (This value is known as the "translucent limit".)

Hint: For the case of zero angular momentum the Laplacian operator in spherical coordinates takes the form

$$\bigtriangledown^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$$

6. Consider a capacitor consisting of two metal plates with a charge $+Q$ on one plate and $-Q$ on the other. In the gap of the capacitor we have two perfectly harmonic springs attached to the top plate—one with a H atom and the other with a H ion attached to the end of the spring in the gap of the capacitor. The springs are close enough together to allow for the electron on the H atom to tunnel to the H ion and visa versa. Ignore gravity, and suppose that initially each spring is in its ground state.




A. Describe qualitatively what happens as the electron tunnels from the one spring to the other.

B. Show how you would calculate what the probability amplitude is for the two springs to be in their new ground states after an instantaneous jump of the electron from one spring to the other.

(The problem is similar to describing the motion of electrons in the presence of electron-phonon interactions in solids.)

Maybe useful notes—the forms of the first few wavefunctions for a simple harmonic oscillator

$$\psi_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-y^2/2}$$
$$\psi_1 = \left(\frac{\alpha}{\pi}\right)^{1/4} \sqrt{2}y\ e^{-y^2/2}$$
$$\psi_2 = \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}}(2y^2 - 1)e^{-y^2/2}$$
$$\psi_3 = \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{3}}(2y^3 - 3y)e^{-y^2/2}$$

$$\alpha \equiv \frac{m\omega}{\hbar} \qquad y = \sqrt{\alpha}x$$

7. Ultra cold atoms at temperatures of a few nanokelvin can exhibit fascinating large-scale quantum physic that is totally unexpected at room temperatures. Recently, ultra cold Fermi gases of $^6$Li atoms ( density $=10^{14}/\text{cm}^3$) have been studied in laboratories. Standard laser cooling has been applied to cool atoms to about 100 microkelvin followed by the evaporative cooling which can further cool atoms to below 100 nK.

A) Calculate the average speeds of these atoms at room temperatures. Can you check your results using what you know about the air?

B) Estimate the average speed of atoms at 100 microkelvin.

C) Is the average speed equal to zero when T=0? If not, what is it?.

8. At 1 AU from the Sun, the radiation flux (integrated over all wavelengths) striking the Earth is 1379 W/m$^2$. (called the Solar Constant $S_o$). Assuming a mean reflective albedo $A$ (fraction of incident light reflected), then

A. Argue that the instantaneous energy absorbed per unit time will be

$$\frac{dE}{dt} = \frac{\pi R_p^2(1-A)S_o}{r_{AU}^2}$$

where $R_p$ is a planet's radius and $r_{AU}$ is the planet's instantaneous distance from the Sun in AU

B. Compute the energy input rate to the planet moving on an elliptical orbit, but averaged over an orbital period of the planet (in units watts/year), where your final answer gives ALL dependencies on $a$ (semi-major axis) and $e$ (eccentricity). (Recall: $e = f/a$, where $f$ is the distance between the centre of the ellipse and either focus.)

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| mass of an electron | $m_e$ | 0.511 MeV/c$^2$ |
| mass of a neutron | $m_n$ | 940 MeV/c$^2$ |
| mass of a proton | $m_p$ | 938 MeV/c$^2$ |
| mass of the Sun | $M_{sun}$ | $2 \times 10^{30}$ kg |
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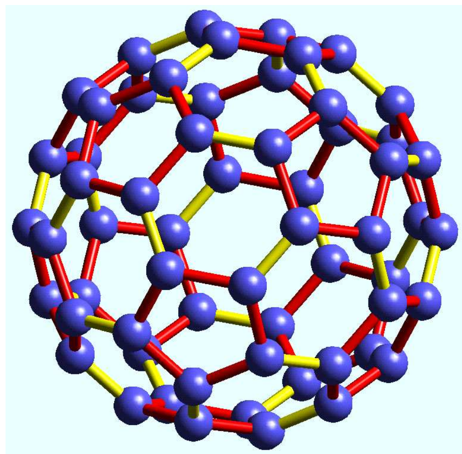
9. A point charge $Q$ with mass $M$ approaches a semi-infinite surface of a perfect metal and becomes trapped near the surface in a (quantum) bound state. Find the binding energy of this particle to the surface. Treat the metal classically—i.e. ignore its internal quantum levels, etc., and just consider it to be a classical perfect conductor that the point charge's wavefunction cannot penetrate.

10.

A. A network of polymers inside a cell support the cell's organelles (internal structures). A researcher wants to measure the stiffness of this network, which acts like a spring constraining the organellese. She therefore embeds a 40 nm diameter bead inside the network. She has the idea that she will measure the effective spring constant $k$ of the network by applying a small force $F$ and relating the force to the resulting displacement by $F = k\Delta x$. She quickly finds that it's really hard to accurately apply and measure such a small force. Instead she just takes a movie of the thermal motion of the bead at room temperature, and finds that the bead's mean-square deviation from its average position is $\sqrt{(\Delta x)^2}$=30 nm. Compute the effective spring constant $k$.

B. The cell's membrane allows sodium ions but not chlorine ions to diffuse through it. The cell is placed in a salty solution with a ten times higher concentration of salt outside than in the cell. As a result sodium diffuses into the cell. Calculate the voltage difference across the cell membrane as a result.

11. What would the spatial distribution of a monochromatic (equal velocity) beam of buckminsterfullerine molecules (chemical formula $C_{60}$), travelling with a speed equal to the RMS speed of molecules in an ideal gas of $C_{60}$ at room temperature, look like after passing through two slits separated by a distance $d$?



12. A solid described as a periodic array of identical atoms, each with on average $n$ valence electrons, can be metallic with electrons tunneling between the atoms if the average occupation number $n$ is between 0 and $M$ where $M$ is the degeneracy of the valence orbital under consideration.

A. What are the $M$ values for $p$, $d$, and $f$-like valence shell?

B. Determine the probability distribution that a particular atom has $L$ valence electrons assuming an independent particle picture, i.e. no interactions between the particles but remembering that the electrons are fermions.

C. Now calculate the energy as a function of $L$ if the electrons on an atom have a coulomb repulsion between each other equal to the binding energy of a H 1s electron.

13. Calculate the minimum number of atoms that must make up an object for gravity to become the dominant force in determining its structure if this means:

A. The object is forced to have a spherical shape. (This condition defines the boundary between an asteroid, which may have an irregular shape, and a dwarf planet, which must be spherical.)

B. Gravity dominates over internal electrostatic forces. (Below this mass, the radius of the object increases with its mass, but above it the radius stops growing significantly with mass. This is partly responsible for the dividing line between a gas giant and a star.)

To help you, note that the compressive strength of most matter is $\sim 10$ MPa.

14. Use the virial theorem to estimate the thermal energy in the Sun (assume that the Sun has uniform density). Hence estimate the Sun's Helmholtz time - that is the time it would take to cool by a factor of two in temperature if the nuclear energy release were to cease.

15. Greenhouse gases such as $CO_2$ and $H_2O$ absorb radiation from the Earth and re-emit back to the Earth, thus contributing to global warming. Carry out an analysis about these molecules to argue or prove that these molecules can indeed absorb the radiation from the Earth. (Only an order of magnitude estimate is needed!)

$CO_2$: Mass=44 a.u., radius=0.12 nm

$H_20$: Mass=18 a.u., radius=0.15 nm.

16.

A. Two protons experience a mutual electrostatic repulsion, and also have a very strong "contact" nuclear interaction that binds the protons together if they actually are brought close enough together to touch, and is negligible otherwise. The diameter of a proton is $10^{-15}$ m. First, draw the interaction energy as a function of the distance between the protons.

B. Next, estimate the approximate temperature of a gas of protons in which the average proton has enough kinetic energy to overcome the electrostatic barrier and bind with another proton.

C. The Sun's actual temperature is quite a bit lower than naive calculation of Part B would suggest, yet it produces energy by fusion. Give two reasons why fusion can still take place even if the temperature of the proton gas is well below the temperature at which the average kinetic energy of a proton exceeds the Coulomb barrier.