Name: ______________ Student #: ______________

PHYS350: Applications of Classical Mechanics
Homework Assignment #2 (Part 2)
Due: Monday, February 25, 2019 (by 9AM)

Problem Worksheet A

Name: ____________ Student #: ____________

Problem 1: Free-Fall of a Dark-Matter Particle through the Earth

Background: Dark matter is elusive stuff! In fact, most dark-matter particles that come our way will pass right through us and other ordinary matter without interacting with it. However, occasionally, a dark-matter particle might scatter from ordinary atoms and this problem is about the fate of such a dark-matter particle.

A dark matter particle of mass m scatters from an Iron nucleus in the crust of the Earth and is momentarily at rest on the surface of the Earth (a distance R from the centre of the Earth) at time $t = 0$. The particle then moves only under the influence of the force of gravity. Approximate the Earth as a sphere of uniform density ρ.

Hint: You may assume that the force exerted on the particle at a distance r from the centre of the Earth is directed towards the centre of the Earth, is proportional to $M(r)$, the mass enclosed at a distance r from the center of the Earth, and the the mass exterior to r exerts no net force on the particle.

(a) Write down an expression for the position-dependent gravitational force $\vec{F}_G(\vec{r})$ on a particle inside of the Earth. Using the fact that $|\vec{F}_G(R)| = mg$, re-express this force in terms of g (the acceleration due to gravity on the surface of the Earth).

(b) Write down Newton's second law for the motion of this particle. Choose a convenient coordinate system and solve this differential equation to find the trajectory $\vec{r}(t)$ with the initial conditions described above (namely, $\dot{\vec{r}}(0) = 0$ and $|\vec{r}(0)| = R$).

(c) Given that $R = 6.37 \times 10^6$ m and $g = 9.81\,\mathrm{m\,s^{-2}}$ find the velocity of the particle as it passes through the centre of the Earth. How many hours will this dark-matter particle take to return back to its starting position?

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Problem 2: A Simple Pendulum

A simple pendulum of mass m and length ℓ moves under the acceleration due to gravity g on the surface of the earth. The pendulum makes an angle θ with respect to the vertical axis.

(a) Write down an expression for the gravitational potential energy U of this pendulum in terms of the height h of the pendulum above the surface of the earth.

(b) Given that the pivot point of the pendulum is a height H above the surface of the earth, determine the potential energy as a function of the arc length $s = \ell\theta$ from equilibrium. Choose the arbitrary zero of potential energy such that $U(0) = 0$

(c) Sketch the potential energy of the pendulum. If the pendulum starts at $s = 0$ with a kinetic energy $T = mg\ell$ what is the allowed range of motion?

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Problem 3: The Point Grey Pendulum

A rod is fixed at a 45° angle to the floor and ceiling of a room. A mass M is constrained to slide along the rod without friction and is attached to a spring with equilibrium length ℓ_s and spring constant k. A pendulum of mass m and length ℓ_p is hung from the mass M.

(a) Assuming the pendulum is constrained to move within the plane containing the rod, choose an appropriate set of generalized coordinates and write down the Lagrangian in this case.

(b) Assuming the pendulum is constrained to perpendicular to the plane containing the rod, choose an appropriate set of generalized coordinates and write down the Lagrangian in this case.

(c) Write down the quadratic Lagrangian close to the stable equilibrium point of the system for both case (a) and (b) above. Which configuration has a simpler description when the pendulum is near equilibrium and why?

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Problem 4: Charged Pendulum on a Charged Wire

A charged pendulum of mass m and length ℓ hangs from a uniformly charged wire suspended at height ℓ. The electric field of the wire exerts a repulsive or attractive force on the pendulum bob depending only on the distance d from the wire of the form $U_{\text{wire}} = mC\ln(d/\ell)$.

(a) Sketch a diagram of this system. How many degrees of freedom does it have? Choose an appropriate generalized coordinate (or set of coordinates) to describe the system.

(b) Determine the total potential energy of the system (including the gravitational potential) and write it in terms of your chosen coordinate. Re-express the total kinetic energy of the system in terms of your chosen generalized coordinate. Write down the Lagrangian for this system.

(c) Determine the generalized momenta and generalized forces of this system. Write down Lagrange's equations for this system.

(d) At some point the system is at rest and the angle θ the pendulum makes from vertical is small ($\theta \ll 1$). Determine the equations of motion close to this equilibrium point and the frequency of small oscillations about this point. What is the condition that this equilibrium point is stable?

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Problem 5: Pendulum in an Elevator

A pendulum of mass m and length ℓ hangs from the ceiling of an elevator. The elevator accelerates uniformly upward with acceleration a.

(a) Sketch a diagram of this system. How many degrees of freedom does it have? Choose an appropriate generalized coordinate (or set of coordinates) to describe the system.

(b) Determine the total potential energy of the system (including the gravitational potential) and write it in terms of your chosen coordinate. Re-express the total kinetic energy of the system in terms of your chosen generalized coordinate. Write down the Lagrangian for this system.

(c) Find the Lagrange equation of motion for this system and show that it is identical to that for a non-accelerating pendulum with g replaced by $g + a$. Find the frequency of small oscillations about the equilibrium point.

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Problem 6: Pendulum on a Wheel

A pendulum of mass m and length ℓ hangs from a point attached to wheel (see Figure 7.14 of Taylor). The wheel uniformly rotates at angular speed ω.

(a) Sketch a diagram of this system. How many degrees of freedom does it have? Choose an appropriate generalized coordinate (or set of coordinates) to describe the system.

(b) Determine the total potential energy of the system (including the gravitational potential) and write it in terms of your chosen coordinate. Re-express the total kinetic energy of the system in terms of your chosen generalized coordinate. Write down the Lagrangian for this system.

(c) Find the Lagrange equation of motion for this system.

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