Calculus of Variations

Problem:

(a) Write down a functional

$$S[y(x)] = \int_{x_0}^{x_1} dx\, f(y, y', x)$$

equal to the total distance between two points (x_0, y_0) and (x_1, y_1) along the curve $y(x)$ and use the method of variations $[y(x) \to y(x) + \alpha\eta(x)]$ to find the differential equation that $y(x)$ must satisfy such that $S[y(x)]$ is extremal. Solve this equation and show that $y(x)$ must be the unique straight line passing through the two points.

(b) Imagine that the integrand $f(y, y', x)$ were left unspecified — corresponding to a different optimization problem than the total distance problem from part (a). Following the same steps and using the same method of variations what differential equation would you arrive at that $y(x)$ must satisfy such that $S[y(x)]$ is extremal?