Name: ____________ Student #: ____________

Applications of Classical Mechanics
Physics 350 2018W
Challenge Problem: Monday, April 1, 2019

Problem: Charged Particle in an Electromagnetic Field

The Lagrangian for a charged particle in an electromagnetic field takes the form

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{\mathbf{r}}^2 - q\phi(\mathbf{r}, t) + q\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, \mathbf{t})$$

where $\phi = \phi(\mathbf{r}, t)$ and $\mathbf{A} = \mathbf{A}(\mathbf{r}, \mathbf{t})$ are the scalar and vector potentials respectively. The corresponding electric and magnetic fields are $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$.

(a) Using the general definition find the Hamiltonian of this system in terms $\mathbf{r}$ and $\dot{\mathbf{r}}$. Show it is equal to $H = T + U$ where T is the kinetic energy of the particle and $U = q\phi$ is the scalar potential energy. Why does the vector potential $\mathbf{A}$ not appear?

(b) Find the canonical momentum $\mathbf{p}$ of this system. Find the canonical form the Hamiltonian $H = H(\mathbf{r}, \mathbf{p})$.

(c) A charged particle moves in a region with a uniform magnetic field $\mathbf{B} = B\hat{k}$ and zero electric field. Show that $\mathbf{A} = Bx\hat{y}$ is one choice that generates this magnetic field. Find the Hamiltonian for this special case. Which coordinates are ignorable?

(d) Find Hamilton's equations for the coordinates x and y.

(e) Show that the value of p_y can be absorbed into a shift in the origin of x (by defining a new coordinate $\tilde{x} = x - p_y/B = x - x_0$). Show that $\frac{d}{dt}(p_x - By) = 0$, and thus $p_x = B\tilde{y}$ for some $\tilde{y} = y - y_0$. Find the general solution for $x(t)$ and $y(t)$.
