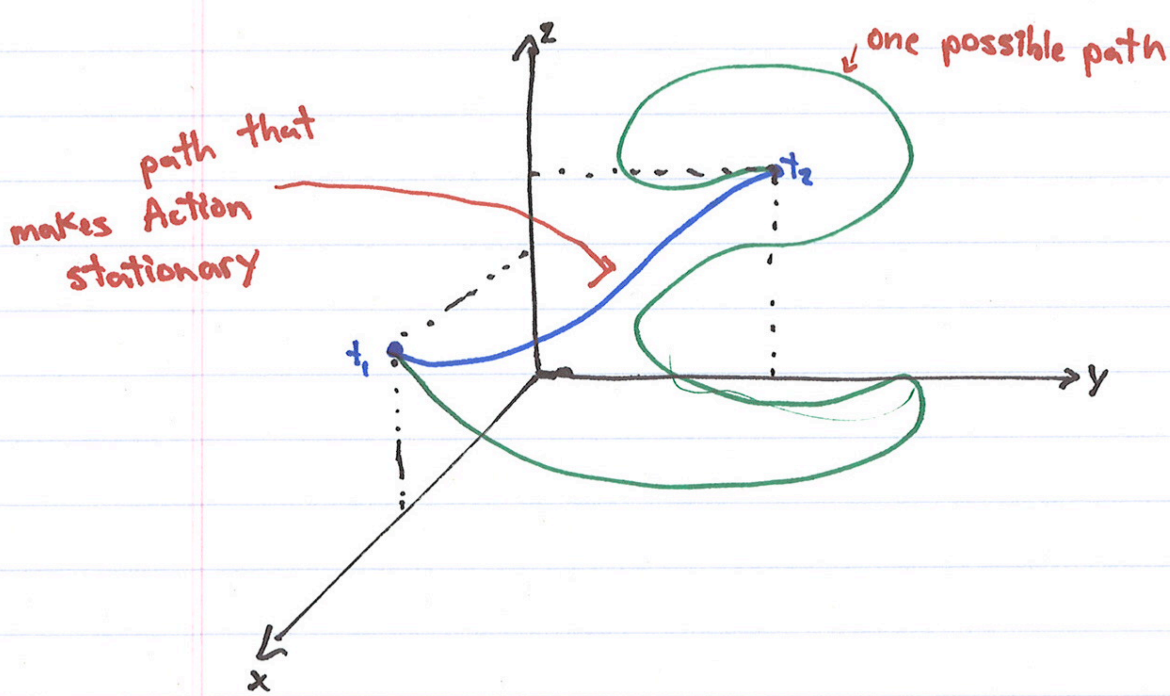
Lagrangian Mechanics

Lagrangian Mechanics describes the dynamics of mechanical systems using the framework of variational calculus.

In particular, according to Hamilton's Principle, the actual motion that a particle follows between time t_1 and t_2 is the motion that makes the Action Integral

$$S = \int_{t_1}^{t_2} dt\, L \qquad \text{stationary.}$$



Lagrangian Mechanics

To specify the trajectory, or dynamical path, of a single particle we generally requre 3 generalized coordinates (q_1, q_2, q_3).

i.e. (x, y, z) or (r, θ, ϕ) etc...

The functions $q_1(t)$, $q_2(t)$, $q_3(t)$ characterize the possible or "potential" trajectories.

The Action is a functional of these functions

$$S[q_1(t), q_2(t), q_3(t)] = \int_{t_1}^{t_2} dt\, L(q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3, t)$$

which leads to the Lagrange Equations

$$\frac{\partial L}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)$$

$$\frac{\partial L}{\partial q_2} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)$$

$$\frac{\partial L}{\partial q_3} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_3}\right)$$

Lagrangian Mechanics

For a single unconstrained particle with Kinetic Energy T, where

$$T = T(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$$

and Potential Energy U, where

$$U = U(q_1, q_2, q_3)$$

The Lagrangian is the difference

$$L = T - U = L(q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3)$$

and the trajectory of the particle satisfies Lagrange's equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad i = 1, 2, 3$$

Lagrangian Mechanics

In cartesian coordinates $q_1 = x$

$q_2 = y$

$q_3 = z$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$U = U(x, y, z)$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

Now...

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x \quad \text{Force}$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x \quad \text{Momentum}$$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \quad \text{Lagrange's Equation}$$

$$\Downarrow$$

$$F_x = \frac{d}{dt}(p_x)$$

$$F_x = \dot{p}_x \quad \text{Newton's Law}$$

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) \Rightarrow F_y = \dot{p}_y$$

$$\frac{\partial L}{\partial z} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) \Rightarrow F_z = \dot{p}_z$$

Lagrangian Mechanics

Note however that Lagrange's equations take the same form in every and any coordinate system we choose to write down

In the general case with Generalized Coordinates

$$L = L(q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3)$$

$$\frac{\partial L}{\partial q_i} \equiv \tilde{F}_{q_i} \quad \text{Generalized Force}$$

$$\frac{\partial L}{\partial \dot{q}_i} \equiv \tilde{p}_{q_i} \quad \text{Generalized Momentum}$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

$$\Downarrow$$

$$\tilde{F}_{q_i} = \dot{\tilde{p}}_{q_i}$$

Lagrangian Mechanics

Example: Polar Coordinates $q_1 = r$ $q_2 = \phi$ $q_3 = 0$

$$T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2)$$
$$U = U(r,\phi)$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r,\phi)$$

The r-coordinate equation:

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}(m\dot{r})$$

$$\therefore \quad mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = m\ddot{r}$$

$$F_r = m(\ddot{r} - r\dot{\phi}^2)$$

The ϕ-coordinate equation:

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$$

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}(mr^2\dot{\phi})$$

Lagrangian Mechanics

Since $F_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi}$

and $\underbrace{\Gamma_\phi}_{\text{Torque}} = r F_\phi$

$\underbrace{L_\phi}_{\text{Angular Momentum}} = m r^2 \dot{\phi}$

we have

$$\Gamma_\phi = \frac{dL_\phi}{dt}$$

Lagrangian Mechanics cont.

Single Unconstrained Particle

Need 3 $\begin{cases} \text{generalized coordinates} & q_1, q_2, q_3 \\ \text{generalized velocities} & \dot{q}_1, \dot{q}_2, \dot{q}_3 \end{cases}$

to specify the state of the system

From these we can construct the..

Kinetic Energy $\quad T = T(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$

Potential Energy $\quad U = U(q_1, q_2, q_3, t)$

Lagrangian $\quad L = T - U = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)$

Action $\quad S = \int_{t_1}^{t_2} dt\, L \quad$ stationary w.r.t variations of $q_i(t)$

The Path Followed By The System Satisfies

Lagrange's Equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad i=1,2,3$$

for ANY choice of q_1, q_2, q_3