Functionals of Many Functions

In analogy with a function of one variable we have discussed a functional of one function.

Function $\qquad\qquad$ Functional

$$s(y_1) \longleftrightarrow S[y(x)]$$

where usually $S[y(x)] = \int_{x_1}^{x_2} dx\, f(y(x), y'(x), x)$

and the E.-L. equations are

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

In analogy with a function of two variables we can also have a functional of two functions.

$$s(y_1, y_2) \longleftrightarrow S[y_1(x), y_2(x)]$$

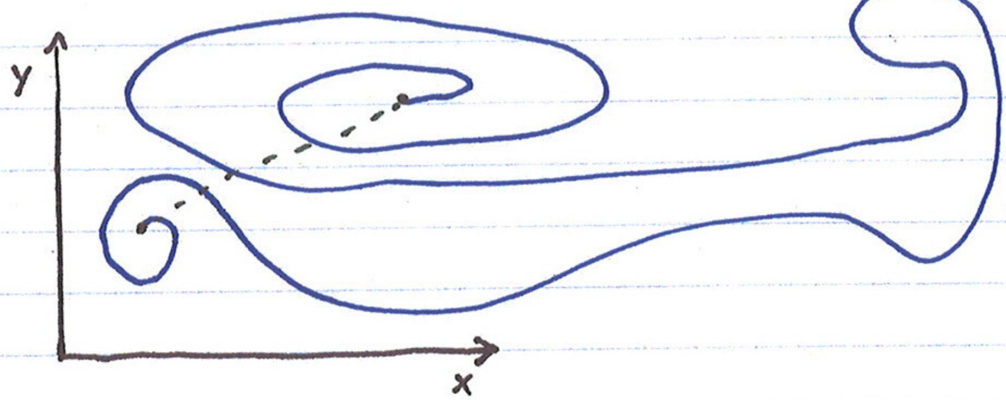
where usually $S[y_1(x), y_2(x)] = \int_{x_1}^{x_2} dx\, f(y_1(x), y_1'(x), y_2(x), y_2'(x), x)$

and $\quad \frac{\partial f}{\partial y_1} = \frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right) \; ; \; \frac{\partial f}{\partial y_2} = \frac{d}{dx}\left(\frac{\partial f}{\partial y_2'}\right)$

two E.-L. equations

Functionals of Many Functions

Example: Shortest Path Between Two Points (Again!)



More generally we can specify two functions $x(u)$ and $y(u)$

$$S[x(u), y(u)] = \int_{u_1}^{u_2} du\, f(x, x', y, y', u) = \int_{u_1}^{u_2} du \sqrt{(x')^2 + (y')^2}$$

$$\frac{d}{du}\left(\frac{\partial f}{\partial x'}\right) = \frac{\partial f}{\partial x} = 0 = \qquad \frac{\partial f}{\partial y} = \frac{d}{du}\left(\frac{\partial f}{\partial y'}\right)$$

$$\Rightarrow \quad \frac{x'}{\sqrt{(x')^2 + (y')^2}} = c_1 \qquad \frac{y'}{\sqrt{(x')^2 + (y')^2}} = c_2$$

$$\Rightarrow \quad \frac{y'}{x'} = \frac{dy}{dx} = \frac{c_1}{c_2}$$

$$\Rightarrow \quad y = mx + b$$

Functionals of Many Functions

In analogy with a function of N variables we can have a functional of N functions

$$\overset{\text{Function}}{S(y_1, y_2, \ldots, y_N)} \longleftrightarrow \overset{\text{Functional}}{S[y_1(x), y_2(x), \ldots, y_N(x)]}$$

where $S[y_1(x), y_2(x), \ldots, y_N(x)] = \int_{x_1}^{x_2} dx\, f(\overbrace{y_1, y_1', y_2, y_2', \ldots, y_N, y_N', x}^{2N+1})$

and

$$\left.\begin{aligned} \frac{\partial f}{\partial y_1} &= \frac{d}{dx}\left(\frac{\partial f}{\partial y_1'}\right) \\ \frac{\partial f}{\partial y_2} &= \frac{d}{dx}\left(\frac{\partial f}{\partial y_2'}\right) \\ &\vdots \\ \frac{\partial f}{\partial y_N} &= \frac{d}{dx}\left(\frac{\partial f}{\partial y_N'}\right) \end{aligned}\right\} \text{N E.-L. Equations}$$

In appications to Lagrangian Mechanics we have

$x \longrightarrow t$ ← time is independent variable/parameter

$y_1(x) \longrightarrow q_1(t)$ ← generalized coordinates

$f(y_1, y_1', y_2, y_2', x) \longrightarrow L(q_1, \dot{q}_1, q_2, \dot{q}_2, t)$

L ← Lagrangian