Calculus of Variations

The Euler-Lagrange Equations

Consider a functional of the form

$$S[y(x)] = \int_{x_1}^{x_2} dx\, f(y(x), y'(x), x)$$

which we imagine is extremized at $y(x)$.

To determine the form of $y(x)$ we consider arbitrary variations about $y(x)$ of the form $\quad Y(x) = y(x) + \alpha\,\eta(x)$

For a variation with a fixed shape $\eta(x)$ we can then consider the functional

$$S[Y(x)] = S[y(x) + \alpha\,\eta(x)] = S(\alpha)$$

to be a function of the parameter α

Expanding about $\alpha = 0$

$$S(\alpha) = \underbrace{S(0)}_{S_{min}} + \left.\frac{\partial S}{\partial \alpha}\right|_0 \alpha + \cdots$$

Calculus of Variations

A necessary condition for $S[y(x)]$ to be extremal is

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = 0$$

Now $S(\alpha) = \int_{x_1}^{x_2} dx\, f(y(x)+\alpha\eta(x), y'(x)+\alpha\eta'(x), x)$

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = \int_{x_1}^{x_2} dx \left(\left.\frac{\partial Y}{\partial \alpha}\right|_0 \left.\frac{\partial f}{\partial Y}\right|_0 + \left.\frac{\partial Y'}{\partial \alpha}\right|_0 \left.\frac{\partial f}{\partial Y'}\right|_0 \right)$$

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = \int_{x_1}^{x_2} dx \left(\eta(x)\frac{\partial f}{\partial y} + \eta'(x)\frac{\partial f}{\partial y'} \right)$$

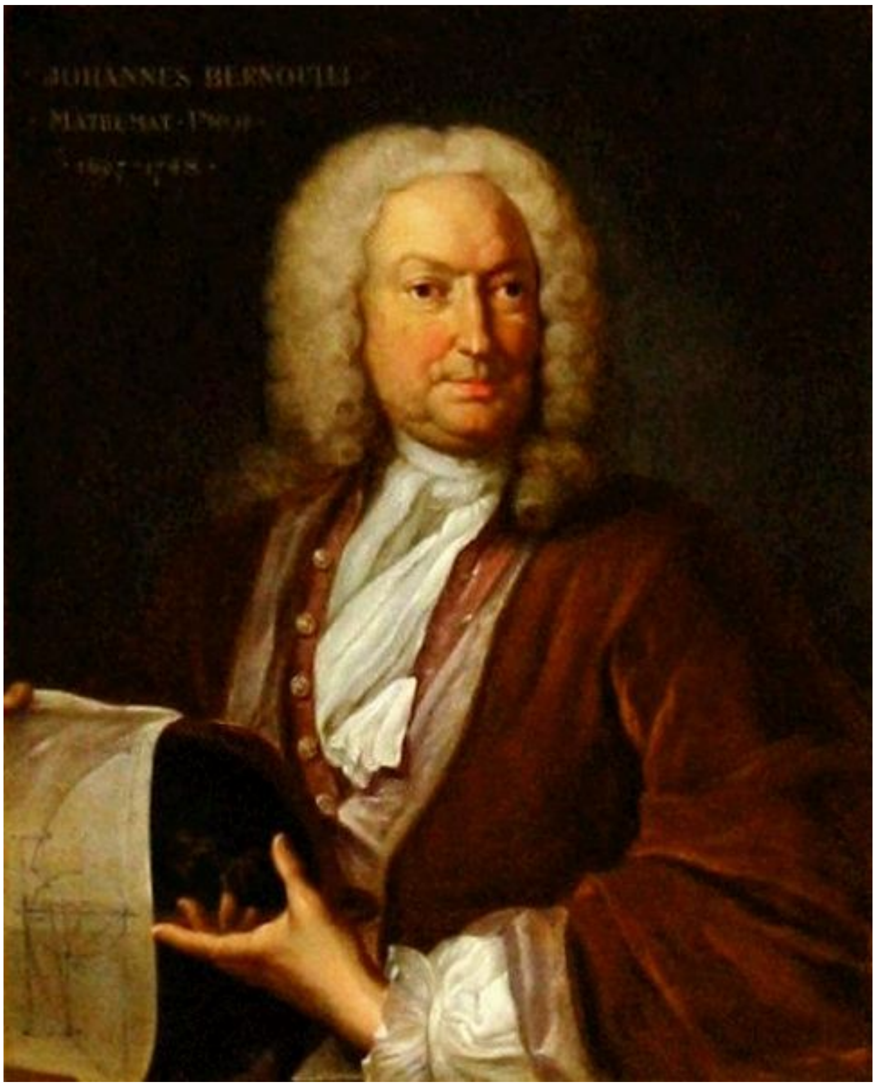
$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = \int_{x_1}^{x_2} dx \left(\eta(x)\frac{\partial f}{\partial y} - \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right) + \left[\eta(x)\frac{\partial f}{\partial y'} \right]_{x_1}^{x_2}$$

but $\left[\eta(x)\frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} = 0$ since $\eta(x_1) = 0 = \eta(x_2)$

$$\therefore \quad \left.\frac{\partial S}{\partial \alpha}\right|_0 = 0 = \int_{x_1}^{x_2} dx\, \eta(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right]$$

but $\eta(x)$ was a variation of arbitrary shape and to satisfy this equation for all $\eta(x)$ we require

$$\boxed{\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)}$$


JOHANNES BERNOULLI

ACTA ERUDITORUM
ANNO M DCLXXXIV
publicata,
ac
SERENISSIMO FRATRUM PARI,
DN. JOHANNI
GEORGIO IV,
Electoratus Saxonici Hæredi,
&
DN. FRIDERICO
AUGUSTO,
Ducibus Saxoniæ &c. &c. &c.
PRINCIPIBUS JUVENTUTIS
dicata,

*Cum S. Cæsareæ Majestatis & Potentissimi
Electoris Saxoniæ Privilegiis.*

LIPSIÆ,
Prostant apud J. GROSSIUM & J. F. GLETITSCHIUM.
Typis CHRISTOPHORI GüNTHERI.
Anno M DCLXXXIV.

”I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.”

Problem: Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.

Calculus of Variations

Example: The Brachistochrone

Question: If a bead is constrained to a wire that passes through points 1 and 2, what shape should the wire have to move from 1 to 2 in the shortest time?

By the conservation of energy

$$E = \frac{1}{2}mv^2 + mg(-y) = 0$$

$$\Rightarrow \quad v = v(y) = \sqrt{2gy}$$

Time: $$T_{1\to 2} = \int_1^2 \frac{ds}{v}$$

We need to write this time as (or interpret it) a functional of the shape of the curve.

Calculus of Variations cont.

Example: The Brachistochrone cont.

As we know $v(y)$ it is easiest to treat y as the independent variable and determine $x(y)$

so...
$$S[x(y)] = \int_0^{y_2} dy \frac{ds}{dy} \frac{1}{\sqrt{2g}} \frac{1}{\sqrt{y}}$$

Now $ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(y)^2 + 1}\, dy$

$$\Rightarrow \quad S[x(y)] = \frac{1}{\sqrt{2g}} \int_0^{y_2} dy \frac{\sqrt{(x'(y))^2 + 1}}{\sqrt{y}}$$

So
$$f(x(y), x'(y), y) = \frac{\sqrt{(x')^2 + 1}}{\sqrt{y}}$$

Euler-Lagrange $\Rightarrow$
$$\frac{\partial f}{\partial x} = \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right)$$

$$\Rightarrow \quad \frac{(x')^2}{y(1+(x')^2)} = \text{const} = \frac{1}{2a}$$

$$x' = \sqrt{\frac{y}{2a - y}}$$

The Brachistochrone cont

Solution:

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

Cycloid