Kinetic Energy and Work

For a single particle the Kinetic Energy is

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

The time derivative is just

$$\frac{dT}{dt} = \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v}) = m \dot{\vec{v}} \cdot \vec{v}$$

but since $\vec{F} = m\dot{\vec{v}} = \dot{\vec{p}}$

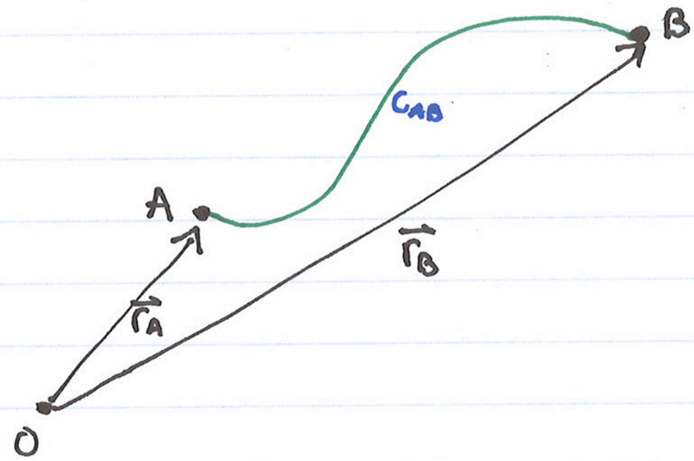
$$\frac{dT}{dt} = \vec{F} \cdot \vec{v} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

Infinitesimally $\quad dT = \underbrace{\vec{F} \cdot d\vec{r}}$

Work done by $\vec{F}$ in displacement $d\vec{r}$

Kinetic Energy and Work

Integrating



$$T_B - T_A = \int_{C_{AB}} \vec{F} \cdot d\vec{r} = W(A \xrightarrow[C_{AB}]{} B)$$

Note: The Change in Kinetic Energy (may) depends upon the path followed!

Conservative Forces

If Force only a function of $\vec{r}$ and can be written as

$$\vec{F} = -\vec{\nabla} U(\vec{r})$$

then we call this force a conservative force.

Here $U(\vec{r})$ is the Potential Energy function.

Example: Gravity

$$\vec{F}_G = -\frac{GMm}{|\vec{r}^3|}\vec{r} = -\frac{GMm}{|\vec{r}|^2}\hat{r}$$

$$U(\vec{r}) \propto \frac{1}{r} \rightarrow U(r) = \frac{A}{r}$$

$$\vec{\nabla} U(r) = -\frac{A}{r^2}\hat{r} \qquad -\vec{\nabla} U(r) = \frac{A}{r^2}\hat{r} = -\frac{GMm}{r^2}\hat{r}$$

$$\therefore \quad U(r) = -\frac{GMm}{r} = -\frac{GMm}{|\vec{r}|}$$

Conservative Forces cont.

$$\Delta T = T_B - T_A = \int_{C_{AB}} \vec{F}\cdot d\vec{r} = W(A \underset{C_{AB}}{\rightarrow} B)$$

For a conservative force we have

$$\vec{F} = -\vec{\nabla} U(\vec{r})$$

$$\Delta T = -\int_{C_{AB}} \vec{\nabla} U(\vec{r}) \cdot d\vec{r}$$

In cartesian coordinates $\quad d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$$

$$\vec{\nabla} = \partial_x\,\hat{i} + \partial_y\,\hat{j} + \partial_z\,\hat{k} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

partial derivatives

$$\vec{F} = \left(-\frac{\partial U}{\partial x}\right)\hat{i} + \left(-\frac{\partial U}{\partial y}\right)\hat{j} + \left(-\frac{\partial U}{\partial z}\right)\hat{k}$$

$$\Delta T = +\int_{C_{AB}} \left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) = \int_{C_{AB}} dU$$

total derivative

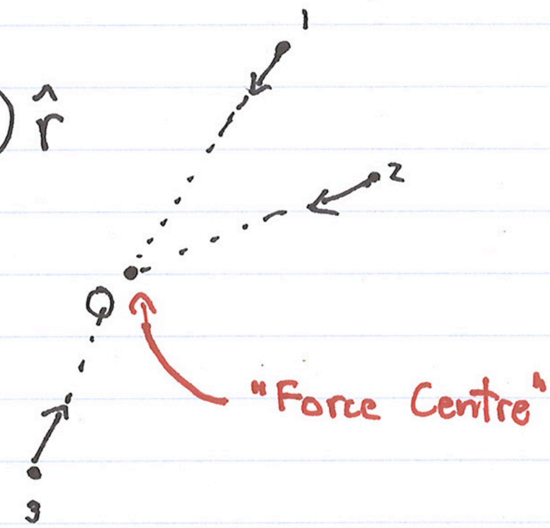
$$\Delta T = U(\vec{r}_B) - U(\vec{r}_A) = W(A \underset{\cancel{C_{AB}}}{\rightarrow} B)$$

Central Forces

Generally a central force takes the form

$$\vec{F}(\vec{r}) = f(\vec{r})\,\hat{r}$$

While the magnitude of the force may depend on $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ the direction is always $\hat{r}$



Central Force
$\vec{F} = f(\vec{r})\,\hat{r}$

$\vec{L}$ conserved
Angular Momentum

E conserved
Energy

Conservative Force
$\vec{F} = -\vec{\nabla} U(\vec{r})$

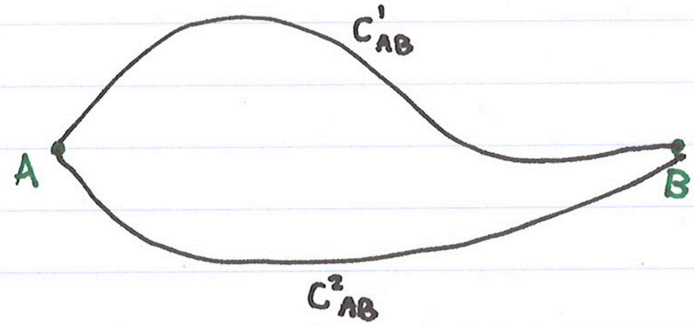
Spherically Symmetric
$\vec{F} = f(r)\,\hat{r} = -\frac{\partial U}{\partial r}\hat{r}$
$U = U(r)$

Single Particle Energetics

Kinetic Energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}}$
(KE)

Work-KE Theorem:

$$\Delta T = T_B - T_A = \int_{A,\ C_{AB}}^{B} \vec{F}(\vec{r})\cdot d\vec{r} = W(A \xrightarrow[C_{AB}]{} B)$$



The change in the Kinetic Energy as a particle moves from position A to position B is the line integral of the net force $\vec{F}$ acting on the particle along a path C_{AB}. This is the work done by $\vec{F}$ along C_{AB}.

In General $W(A \xrightarrow[C^1_{AB}]{} B) \neq W(A \xrightarrow[C^2_{AB}]{} B)$

Work can be path dependent

Single Particle Energetics cont.

Conservative Forces:

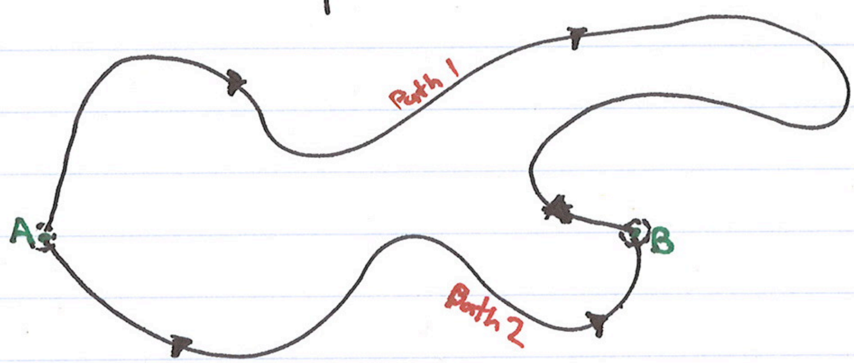
If there exists a scalar function $U(\vec{r})$ such that the force at any position $\vec{r}$ is

$$\vec{F}(\vec{r}) = -\vec{\nabla} U(\vec{r})$$

only defined up to a constant shift $U(\vec{r}) \to U(\vec{r}) + C$

then $\vec{F}$ is a conservative force.

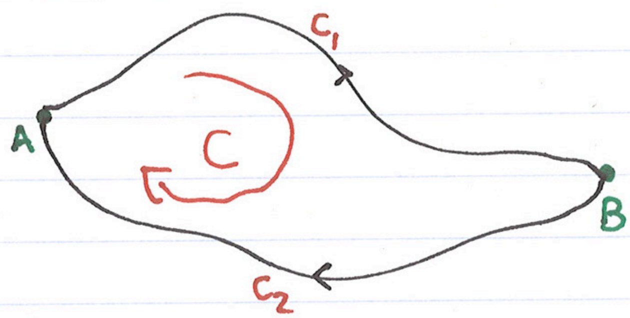
For a conservative force the work done as a particle moves from A to B is Independent of the Path taken between the two points.



$$-\underbrace{\int_A^B \vec{\nabla} U \cdot d\vec{r}}_{\text{Path 1}} = -\underbrace{\int_A^B \vec{\nabla} U \cdot d\vec{r}}_{\text{Path 2}}$$

Proof of Path Independence

Consider two distinct arbitrary points A and B, and draw an arbitrary closed curve passing through both points.



$C = C_1 \cup C_2$

For an arbitrary force field $\vec{F}$ we have

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{A\,(C_1)}^{B} \vec{F}\cdot d\vec{r} + \int_{B\,(C_2)}^{A} \vec{F}\cdot d\vec{r} = \int_{A\,(C_1)}^{B} \vec{F}\cdot d\vec{r} - \int_{A\,(C_2)}^{B} \vec{F}\cdot d\vec{r}$$

By Stokes Theorem: $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \iint_\Sigma (\vec{\nabla}\times\vec{F})\cdot d\vec{\Sigma}$

we can write the line integral of $\vec{F}$ as a surface integral of the curl of $\vec{F}$ ($\vec{\nabla}\times\vec{F}$).

$C = \partial\Sigma$ C is the boundary of Σ

Proof of Path Independence cont.

For a conservative force

$$\vec{F} = -\vec{\nabla} U(\vec{r})$$

$$\therefore \quad \vec{\nabla} \times \vec{F} = -\vec{\nabla} \times \vec{\nabla} U = 0$$

"The curl of the gradient of a scalar field vanishes"

$$\therefore \quad \oint_C \vec{F} \cdot d\vec{r} = \iint_\Sigma (\underbrace{\vec{\nabla} \times \vec{F}}_{\text{everywhere } 0}) \cdot d\vec{\Sigma} = 0$$

$$\Rightarrow \quad 0 = \int_{A,\,C_1}^{B} \vec{F} \cdot d\vec{r} - \int_{A,\,C_2}^{B} \vec{F} \cdot d\vec{r}$$

$$\therefore \quad \int_{A,\,C_1}^{B} \vec{F} \cdot d\vec{r} = \int_{A,\,C_2}^{B} \vec{F} \cdot d\vec{r}$$

As C_1 and C_2 are two distinct arbitrary paths between two arbitrary points we have the desired result.

Single Particle Energetics

Potential Energy:

For a conservative force, the force is the gradient of the potential energy function

$$\vec{F} = -\vec{\nabla} U$$

$$U = U(\vec{r})$$

usually only a function of position

We may also write...

arbitrary constant

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,' + U(\vec{r}_0)$$

If there are many forces acting on a particle $\{\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots, \vec{F}_N\}$ which are all conservative we may write

$$\vec{F}_\alpha = -\vec{\nabla} U_\alpha$$

potential energy for the α^{th} force

The Total Potential Energy is $U(\vec{r}) = \sum_\alpha U_\alpha(\vec{r})$

and $\vec{F} = -\vec{\nabla} U = \sum_\alpha \vec{F}_\alpha$

Single Particle Energetics cont.

Total Energy (Total Mechanical Energy)

We have shown previously that the differential change in kinetic energy when a particle moves from $\vec{r}$ to $\vec{r} + d\vec{r}$ is

$$dT = \frac{dT}{dt}dt = (m\dot{\vec{v}}\cdot\vec{v})dt = \vec{F}\cdot d\vec{r}$$

The change in $U(\vec{r})$ is just the total differential

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = \vec{\nabla}U\cdot d\vec{r}$$

If we define $E = T + U$ then

$$dE = (\vec{F} + \vec{\nabla}U)\cdot d\vec{r}$$

or, for conservative forces, $\boxed{dE = 0}$

"The energy is conserved"

E is a constant of motion.

Single Particle Energetics cont.

Time Dependent Potential Energy $\vec{F} = -\vec{\nabla} U(\vec{r},t)$

We assumed when deriving the conservation of energy that the system was independent of time. The choice of the origin of time is irrelevant and if we take

$t \rightarrow t + c$ the physics is the same.

c ← constant

Note: $U(\vec{r},t) \neq U(\vec{r},t+c)$

What if $U = U(\vec{r},t)$?

Then $$dT = \vec{F} \cdot d\vec{r}$$

but $$dU = \vec{\nabla} U \cdot d\vec{r} + \frac{\partial U}{\partial t} dt$$

$$\Rightarrow \quad dE = (\vec{F} + \vec{\nabla} U) \cdot d\vec{r} + \frac{\partial U}{\partial t} dt$$

$$\dot{E} = \frac{\partial U}{\partial t} \neq 0$$

Energy of particle is not constant.

Many Particle Energetics

For a system of N particles labeled
$\alpha = 1, 2, 3, \ldots, N$

Total Kinetic Energy

$$T = \sum_{\alpha} T_{\alpha} = \sum_{\alpha} \frac{1}{2} m_{\alpha} \vec{v}_{\alpha}^{\,2}$$

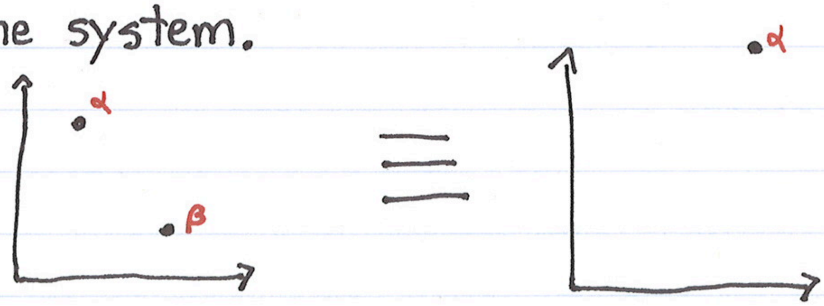
$$T = T(\underbrace{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_N})$$

Function of $3N$ components of velocity. (Actually, only N magnitudes)

Claim: The interaction energy of a pair of particles α and β can be written as

$$U_{\alpha\beta} = U_{\alpha\beta}(\vec{r}_{\alpha} - \vec{r}_{\beta})$$

This follows from the translation invariance of the system.



Many Particle Energetics cont.

The force on particle α or β is just the gradient $\vec{\nabla}_\alpha$ or $\vec{\nabla}_\beta$ with respect to the particle's coordinates (position) $\vec{r}_\alpha$ or $\vec{r}_\beta$ of $U_{\alpha\beta}$

ie. $$\vec{\nabla}_\alpha = \frac{\partial}{\partial x_\alpha}\hat{i} + \frac{\partial}{\partial y_\alpha}\hat{j} + \frac{\partial}{\partial z_\alpha}\hat{k}$$

$$\vec{F}_\alpha = -\vec{\nabla}_\alpha U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta)$$

$$\vec{F}_\beta = -\vec{\nabla}_\beta U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta) = \vec{\nabla}_\alpha U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta)$$

Notice $\vec{F}_\alpha = -\vec{F}_\beta$ by the chain rule!

Newton's third law

In addition the this internal interaction energy each particle may be subject to external conservative forces described by an external potential $U_\alpha^{ext}(\vec{r}_\alpha)$

$\Rightarrow$ 2-Particle Potential: $$U(\vec{r}_\alpha, \vec{r}_\beta) = U_{\alpha\beta}^{int}(\vec{r}_\alpha - \vec{r}_\beta) + U_\alpha^{ext}(\vec{r}_\alpha) + U_\beta^{ext}(\vec{r}_\beta)$$

Many Particle Energetics cont.

Total Potential Energy

$$U = \underbrace{U^{int}}_{\text{Total Internal}} + \underbrace{U^{ext}}_{\text{Total External}} = U(\underbrace{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N}_{\text{Function of 3N components of position.}})$$

$$U^{ext} = \sum_{\alpha} U_{\alpha}^{ext}(\vec{r}_{\alpha})$$

where

$$U^{int} = \underbrace{\sum_{\alpha} \sum_{\beta > \alpha}}_{\text{Sum over distinct pairs}} U_{\alpha\beta}(\vec{r}_{\alpha} - \vec{r}_{\beta})$$

The force on the α^{th} particle is just

$$\vec{F}_{\alpha} = -\vec{\nabla}_{\alpha} U$$

Many Particle Energetics cont.

Total Energy of the System

$$T = \sum_\alpha \frac{1}{2} m_\alpha \vec{v}_\alpha^{\,2}$$

$$dT = \frac{dT}{dt} dt = \sum_\alpha m_\alpha \dot{\vec{v}}_\alpha \cdot \vec{v}_\alpha \, dt$$

$$dT = \sum_\alpha \vec{F}_\alpha \cdot d\vec{r}_\alpha$$

$$U = U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$$

$$dU = \sum_\alpha \vec{\nabla}_\alpha U \cdot d\vec{r}_\alpha$$

$$\Rightarrow \quad dE = d(T+U) = \sum_\alpha (\vec{F}_\alpha + \vec{\nabla}_\alpha U) \cdot d\vec{r}_\alpha$$

$$dE = 0$$

Total Energy $\underline{E} = T + U$ constant

Many Particle Energetics cont.

Rigid Bodies

In general, we have

$$U^{int} = \sum_{\alpha} \sum_{\beta > \alpha} U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta)$$

where for a rigid body made up of atoms $\vec{r}_\alpha$ represent the position of the atoms. In fact, as the interatomic forces are usually 'central we have

$$U^{int} = \sum_{\alpha} \sum_{\beta > \alpha} U_{\alpha\beta}(|\vec{r}_\alpha - \vec{r}_\beta|)$$

U^{int} only a function of the $|\vec{r}_\alpha - \vec{r}_\beta|$

But, by definition, the relative separation of the atoms in a rigid body is constant

$\therefore$ U^{int} is constant for a rigid body and may be ignored

Many Particle Energetics cont.

The Virial Theorem

Relates Average Total Kinetic Energy
and Average Total Potential Energy

The theorem states, for $V(r) = C r^{n+1}$

$$\bar{T} = \frac{n+1}{2}\bar{V}$$

this is what we called U previously

ie. Gravity $\rightarrow$ $n = -2$ $\quad \bar{T} = -\frac{1}{2}\bar{V}$

Harmonic Oscillator $\rightarrow$ $n = 1$ $\quad \bar{T} = \bar{V}$

Note: "n" is NOT number of particles N. "n" is the dependence of the force law on distance. "$n=-2$" $\Leftrightarrow$ $1/r^2$ force

Many Particle Energetics cont.

Proof: The Virial Theorem

Let $$G = \sum_\alpha \vec{p}_\alpha \cdot \vec{r}_\alpha \qquad \alpha = 1, \ldots, N$$

Then $$\frac{dG}{dt} = \sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{p}_\alpha + \sum_\alpha \dot{\vec{p}}_\alpha \cdot \vec{r}_\alpha$$

For the first term notice

$$\sum_\alpha \dot{\vec{r}}_\alpha \cdot \vec{p}_\alpha = \sum m_\alpha \dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha = 2T$$

While for the second term

$$\sum_\alpha \dot{\vec{p}}_\alpha \cdot \vec{r}_\alpha = \underbrace{\sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha}_{\text{Virial of Clausius}}$$

So $$\frac{dG}{dt} = 2T + \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha$$

Lets define the time-average

$$\overline{\frac{dG}{dt}} \equiv \frac{1}{\tau}\int_0^\tau \frac{dG}{dt}\,dt = 2\overline{T} + \overline{\sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha}$$

But $$\overline{\frac{dG}{dt}} = \frac{1}{\tau}\left[G(\tau) - G(0)\right] \xrightarrow[\tau \to \infty]{} 0$$ if $G(\tau)$ bounded (periodic, for instance)

Many Particle Energetics cont.

We thus have

$$\overline{T} = -\frac{1}{2} \overline{\sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha}$$

Conservative Forces

For $\vec{F}_\alpha = -\vec{\nabla}_\alpha U$

$$\overline{T} = +\frac{1}{2} \overline{\sum_\alpha \vec{\nabla}_\alpha U \cdot \vec{r}_\alpha}$$

Now if $U_{\alpha\beta} \propto |\vec{r}_\alpha - \vec{r}_\beta|^{n+1}$ and $U = \sum_\alpha \sum_{\beta>\alpha} U_{\alpha\beta}$

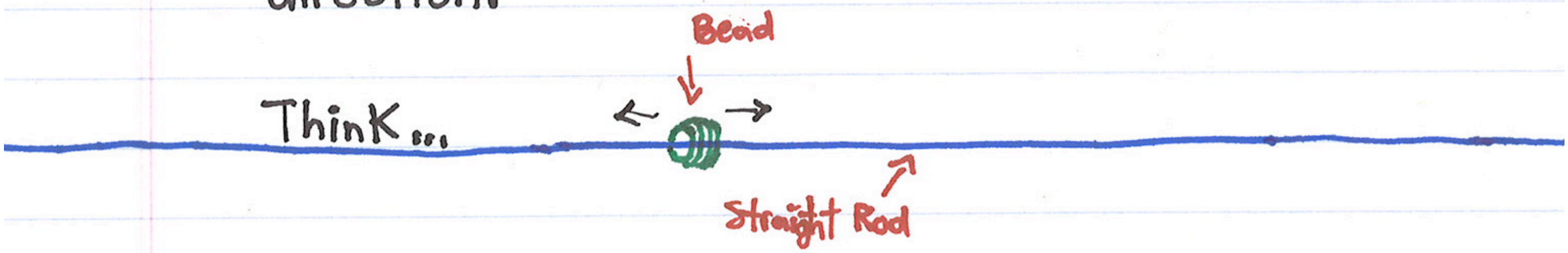
we may show that $\sum_\alpha \vec{\nabla}_\alpha U \cdot \vec{r}_\alpha = (n+1) U$

$$\therefore \overline{T} = \frac{n+1}{2} \overline{U}$$

Motion in One Dimension

Consider a particle (or more generally some body) constrained to move forward or backward along a single direction.

Think...



In general this could be any direction $\hat{n}$, but we may as well choose our coordinate system so that $\hat{n} = \hat{x}$.

The state of this system at some time t is completely characterized by the single coordinate x, which is the position of the particle, and its time derivative $\dot{x}$.

In equations:

$$\left.\begin{aligned} \vec{r}(t) &= x(t)\,\hat{x} \\ \dot{\vec{r}} &= \dot{x}(t)\,\hat{x} \end{aligned}\right\}$$

We can choose $y = z = 0$ WLOG

Motion in One Dimension cont.

In 1D $\quad m\ddot{\vec{r}} = \dot{\vec{p}} = \vec{F}$

simplifies to $\quad m\ddot{x} = \dot{p}_x = F_x$

or dropping subscripts... $\quad m\ddot{x} = \dot{p} = F$

In general we have $F = F(x, \dot{x}, t)$, but lets first consider conservative forces of the form $F = F(x)$

In this case we have

$$F = -\frac{\partial U}{\partial x} = -\frac{dU}{dx}$$

where $U = U(x)$ is the 1D potential energy function.

Motion in One Dimension cont.

As 1D motion in a conservative force field is a special case of 3D motion we know that the

Total energy $$E = T + U$$

is a constant of motion. Where the kinetic energy is just

$$T = \frac{1}{2} m \dot{x}^2 \quad \text{for 1D.}$$

As the kinetic energy is a positive definite quantity we know the motion is subject to the constraint condition

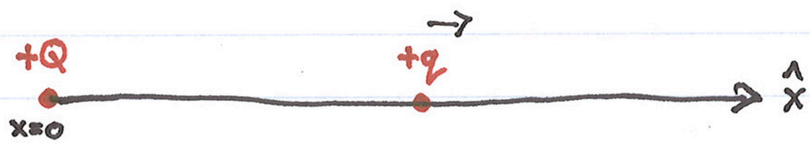
$$E \geq U$$

for any potential $U(x)$.

This knowledge is very powerful.

Motion in One Dimension cont.

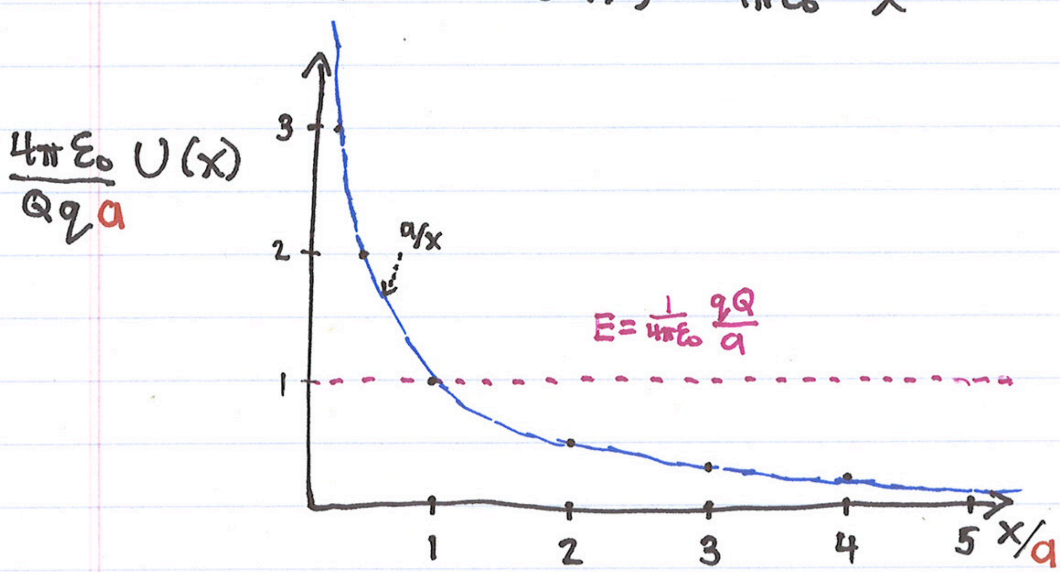
Example: Point charge $+q$ constrained to x-axis in electric field of charge $+Q$ fixed at the origin.



We know $\vec{F}(x) = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{x^2}\hat{x}$

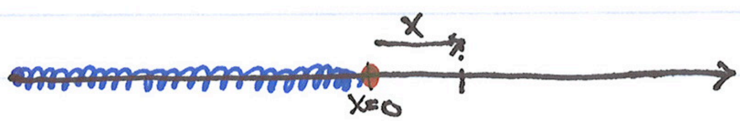
$$\Rightarrow \quad F(x) = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{x^2} = -\frac{\partial U}{\partial x}$$

So $\quad U(x) = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{x}$



Motion in One Dimension cont.

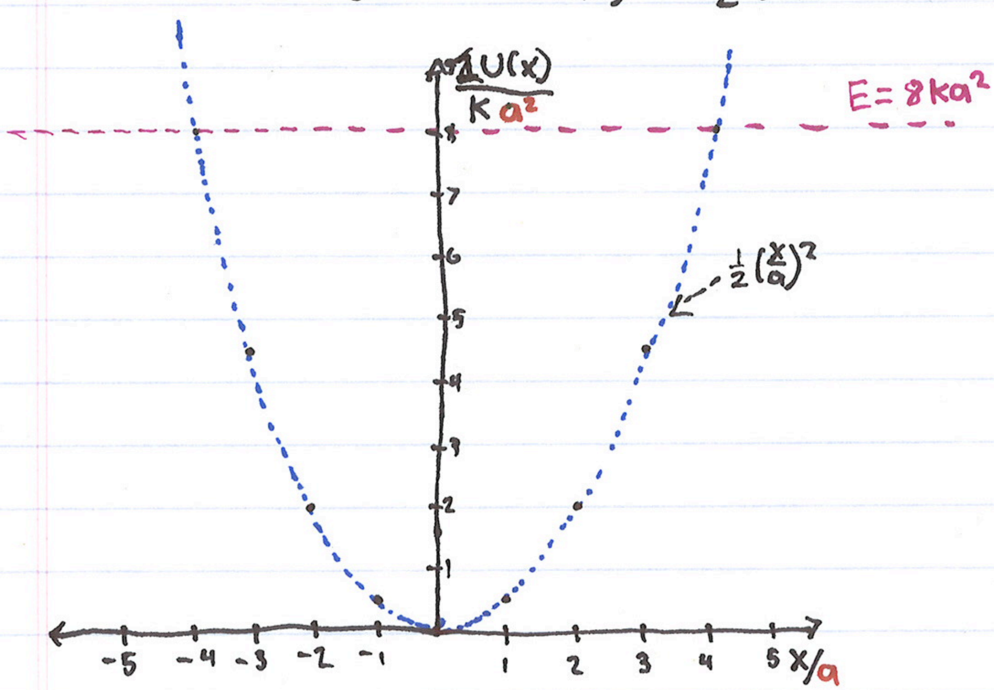
Example: Harmonic Oscillator / Mass on Spring



We Know $\vec{F}(x) = -Kx\hat{x}$

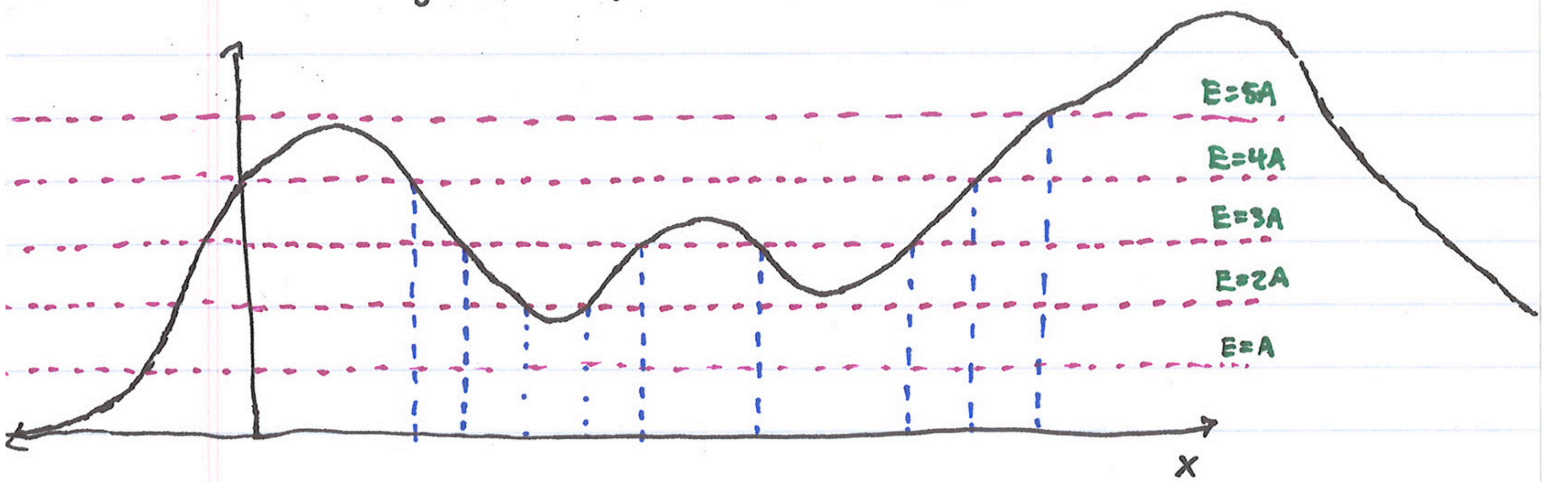
$$\Rightarrow \quad F(x) = -Kx$$

so $U(x) = \frac{1}{2}Kx^2$



Motion in One Dimension cont.

For a general potential $U(x)$



The value of the energy E determines the allowed range of motion.

The set of points x_t such that

$$E = U(x_t) \Rightarrow x_t = U^{-1}(E)$$

are known as the classical turning points of the motion. Why?

Note: If $E < U(x)$ $\forall x$ then $T < 0$
$\Rightarrow$ No solution

Motion in One Dimension cont.

In addition to allowing the determination of the range of motion, the conservation of energy allows us to find a complete solution of the motion for 1D systems.

$$E = \frac{1}{2} m \dot{x}^2 + U(x)$$

$$\Rightarrow \quad \dot{x} = \pm \sqrt{\frac{2}{m}} \sqrt{E - U(x)}$$

choice of sign depends on direction of motion.

Now $\quad dt = \frac{dx}{\dot{x}}$

$$\Rightarrow \quad t(x) - t(x_0) = \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}}$$

If we have $t(x)$ the inverse function is $x(t)$

Notice that two constants E and x_0 describe the motion.

Since $E = \frac{1}{2} m v_0^2 + U(x_0)$ we could have equally used v_0 and x_0