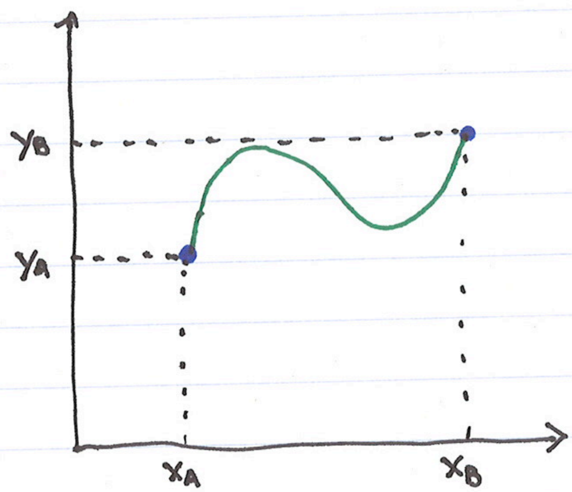
Calculus of Variations

The Shortest Path Between Two Points

Consider two points (x_A, y_A) and (x_B, y_B) in the (x,y) plane



What curve $y(x)$ has the shortest path length S between the endpoints (x_A, y_A) and (x_B, y_B)?

How can we prove this?

Calculus of Variations

The total path length S is

$$S = \int_A^B ds = \int_{x_A}^{x_B} dx \sqrt{1+[y'(x)]^2}$$

since $\quad ds = \sqrt{dx^2+dy^2}$

End Points (x_A, y_A) and (x_B, y_B) are fixed

W.L.O.G we can set $\quad x_A = y_A = 0$
$\quad x_B = \ell \quad y_B = h$

We can choose any path $y(x)$ we like and S depends on the entire function $y(x)$ (actually $y'(x)$ in this case). We should somehow denote this and we write

$$S = S[y(x)] = \int_0^\ell dx \sqrt{1+[y'(x)]^2}$$

S is a **functional** of $y(x)$
(of 1 function)

Calculus of Variations

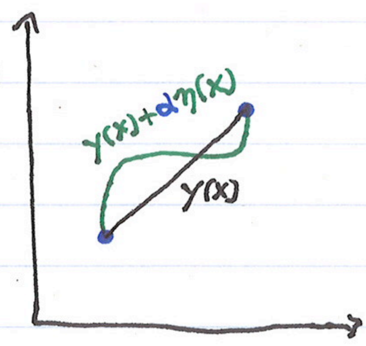
Our goal is to ensure that $y(x)$ minimizes S at $S[y(x)] = S_{min}$

To do this we paramatrize arbitrary variations about the extremal path $y(x)$.

That is, we write $Y(x) = y(x) + \alpha\,\eta(x)$ for arbitrary $\eta(x)$ such that $\underbrace{\eta(x_A) = \eta(x_B) = 0}_{\text{vanishes on the ends}}$

We then consider the functional

$$S[Y(x)] = S[y(x) + \alpha\,\eta(x)]$$



for a variation $\eta(x)$ with fixed shape we can consider $S = S(\alpha)$

$$S(\alpha) = \underbrace{S(0)}_{S_{min}} + \left.\frac{\partial S}{\partial \alpha}\right|_{\alpha=0} \alpha + \ldots$$

A necessary condition for $S[y(x)] = S(\alpha=0) = S_{min}$

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = 0$$

Calculus of Variations

$$S(\alpha) = \int_0^\ell dx \sqrt{1 + [y'(x) + \alpha \eta'(x)]^2}$$

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = \int_0^\ell dx \left(\frac{y'(x)}{\sqrt{1+[y'(x)]^2}} \right) \eta'(x) = 0$$

Using integration by parts

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = \left[\eta(x) \left(\frac{y'(x)}{\sqrt{1+[y'(x)]^2}} \right) \right]_0^\ell - \int_0^\ell dx \frac{d}{dx} \left(\frac{y'(x)}{\sqrt{1+[y'(x)]^2}} \right) \eta(x)$$

$$\Rightarrow \quad - \int_0^\ell dx \frac{d}{dx} \left(\frac{y'(x)}{\sqrt{1+[y'(x)]^2}} \right) \eta(x) = 0$$

arbitrary

$$\therefore \quad \frac{d}{dx} \left(\frac{y'(x)}{\sqrt{1+[y'(x)]^2}} \right) = 0$$

Differential Equation for $y(x)$

Solve with boundary conditions $y(0) = 0$ $y(\ell) = h$

Integrating $\Rightarrow \quad y' = c_1 \sqrt{1 + (y')^2}$ $\quad c_1$ constant

$$\Rightarrow \quad y' = c_2 = \frac{c_1}{\sqrt{1 - c_1^2}} \qquad c_2 \text{ constant}$$

$$\Rightarrow \quad y(x) = c_2 x + c_3$$

$$\Rightarrow \therefore \quad y(x) = \frac{h}{\ell} x$$

Calculus of Variations

A real function $S(y_1, y_2, \ldots, y_N)$ takes as input the variables y_j which have an integer index j

$$S(y_1, y_2, y_3, \ldots, y_N) : \text{Map } \mathbb{R}^N \longrightarrow \mathbb{R}$$

A real functional $S[y(x)]$ takes as input the variables $y(x)$ which have a real index x

$$S[y(x)] : \text{Map } C^1 \longrightarrow \mathbb{R}$$

"Space" of continuos functions

In physics the functionals we are most often concerned with are integrals of the form

$$S[y(x)] = \int_{x_1}^{x_2} dx\, f(y(x), y'(x), x)$$

Where we first consider y and y' independent variables of the integrad $f(y, y'; x)$

Calculus of Variations

The Euler-Lagrange Equations

Consider a functional of the form

$$S[y(x)] = \int_{x_1}^{x_2} dx\, f(y(x), y'(x), x)$$

which we imagine is extremized at $y(x)$.

To determine the form of $y(x)$ we consider arbitrary variations about $y(x)$ of the form $\quad Y(x) = y(x) + \alpha\,\eta(x)$

For a variation with a fixed shape $\eta(x)$ we can then consider the functional

$$S[Y(x)] = S[y(x) + \alpha\,\eta(x)] = S(\alpha)$$

to be a function of the parameter α

Expanding about $\alpha = 0$

$$S(\alpha) = \underbrace{S(0)}_{S_{min}} + \left.\frac{\partial S}{\partial \alpha}\right|_0 \alpha + \cdots$$

Calculus of Variations

A necessary condition for $S[y(x)]$ to be extremal is

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = 0$$

Now $S(\alpha) = \int_{x_1}^{x_2} dx\, f(y(x)+\alpha\eta(x), y'(x)+\alpha\eta'(x), x)$

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = \int_{x_1}^{x_2} dx \left(\left.\frac{\partial y}{\partial \alpha}\right|_0 \left.\frac{\partial f}{\partial y}\right|_0 + \left.\frac{\partial y'}{\partial \alpha}\right|_0 \left.\frac{\partial f}{\partial y'}\right|_0 \right)$$

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = \int_{x_1}^{x_2} dx \left(\eta(x)\frac{\partial f}{\partial y} + \eta'(x)\frac{\partial f}{\partial y'} \right)$$

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = \int_{x_1}^{x_2} dx \left(\eta(x)\frac{\partial f}{\partial y} - \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right) + \left[\eta(x)\frac{\partial f}{\partial y'} \right]_{x_1}^{x_2}$$

but $\left[\eta(x)\frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} = 0$ since $\eta(x_1) = 0 = \eta(x_2)$

$$\therefore \quad \left.\frac{\partial S}{\partial \alpha}\right|_0 = 0 = \int_{x_1}^{x_2} dx\, \eta(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right]$$

but $\eta(x)$ was a variation of arbitrary shape and to satisfy this equation for all $\eta(x)$ we require

$$\boxed{\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)}$$