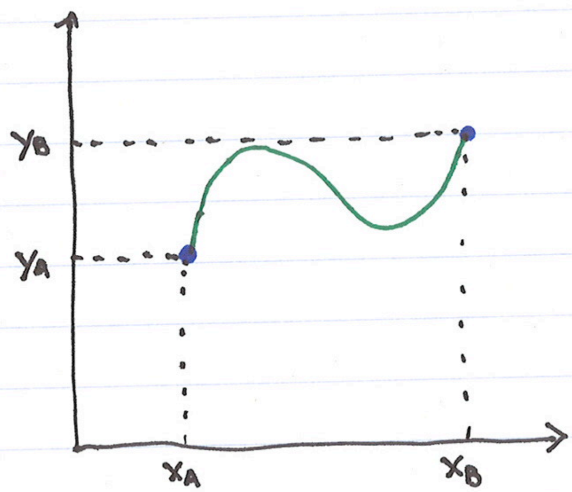
Calculus of Variations

The Shortest Path Between Two Points

Consider two points (x_A, y_A) and (x_B, y_B) in the (x,y) plane



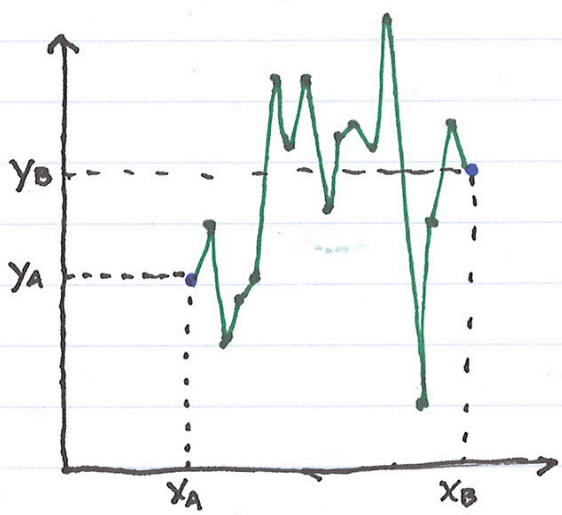
What curve $y(x)$ has the shortest path length S between the endpoints (x_A, y_A) and (x_B, y_B)?

How can we prove this?

Calculus of Variations

The Shortest N+1-segment Path Between Two Points

Consider two points (x_A, y_A) and (x_B, y_B) in the (x,y) plane



What values of $\{y_1, y_2, y_3, \ldots, y_N\}$ lead to the shortest N+1-segment path length S between the end points (x_A, y_A) and (x_B, y_B)?

How can we prove this?

Calculus of Variations

The total path length S in the $N+1$-segment case is

$$S = S_1 + S_2 + \cdots + S_N + S_{N+1} = \sum_{j=1}^{N+1} S_j$$

where $$S_j = \sqrt{(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2}$$

End Points $$(x_0, y_0) = (x_A, y_A)$$
$$(x_{N+1}, y_{N+1}) = (x_B, y_B)$$

W.L.O.G we can set $x_0 = y_0 = 0$
$x_{N+1} = \ell \quad y_{N+1} = h$

We take $$x_j - x_{j-1} = \Delta x = \frac{1}{N+1}\ell \qquad \forall j = \{1, \cdots, N+1\}$$

$$\Rightarrow \quad S = S(y_1, y_2, \cdots, y_N) = \sum_{j=1}^{N+1} \sqrt{\Delta x^2 + (y_j - y_{j-1})^2}$$

S is a function of $y_1, y_2, \cdots, y_N$
(of N variables)

Calculus of Variations

Our goal is to minimize S at $S_{min.}$ Or rather, to ensure that S is minimized at $S(y_1, y_2, \ldots y_N) = S_{min.}$

As we did previously, we are free to choose the minimum at $(y_1, y_2, \ldots, y_N)$ provided we paramatrize arbitrary **variations** away from the minimum.

That is, we write $\quad Y_j = y_j + \alpha \eta_j$

so that $\quad S(Y_1, Y_2, \ldots, Y_N) = S(y_1 + \alpha\eta_1, y_2 + \alpha\eta_2, \ldots, y_N + \alpha\eta_N)$

for a variation $\eta_1, \eta_2, \ldots, \eta_N$ with fixed values we can consider $\quad S = S(\alpha)$

$$S(\alpha) = \underbrace{S(0)}_{S_{min}} + \left.\frac{\partial S}{\partial \alpha}\right|_{\alpha=0} \alpha + \ldots$$

A necessary condition for $S(y_1, y_2, \ldots, y_N) = S(\alpha = 0) = S_{min}$

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = 0$$

Calculus of Variations

$$S(\alpha) = \sum_{j=1}^{N+1} \sqrt{\Delta x^2 + [y_j - y_{j-1} + \alpha(\eta_j - \eta_{j-1})]^2}$$

By the chain rule $\left.\frac{\partial S}{\partial \alpha}\right|_0 = \sum_{i=1}^{N} \left.\frac{\partial S}{\partial Y_i}\right|_0 \frac{\partial Y_i}{\partial \alpha} = \sum_{i=1}^{N} \left.\frac{\partial S}{\partial y_i}\right|_0 \eta_i$

$$\left.\frac{\partial S}{\partial \alpha}\right|_0 = 0 = \sum_{i=1}^{N} \left.\frac{\partial S}{\partial y_i}\right|_0 \eta_i$$

arbitrary

$$\Rightarrow \quad \left.\frac{\partial S}{\partial y_i}\right|_0 = 0 \quad \forall i_{1,\dots,N} \quad \text{individually}$$

N Algebraic Equations for $y_1, y_2, \dots, y_N$

Now $\left.\frac{\partial S}{\partial y_i}\right|_0 = \sum_{j=1}^{N+1} \left[\frac{y_j - y_{j-1}}{\sqrt{\Delta x^2 + (y_j - y_{j-1})^2}} \frac{\partial y_j}{\partial y_i} - \frac{y_j - y_{j-1}}{\sqrt{\Delta x^2 + (y_j - y_{j-1})^2}} \frac{\partial y_{j-1}}{\partial y_i} \right]$

$$= \sum_{j=1}^{N+1} \left[\frac{y_j - y_{j-1}}{\sqrt{\Delta x^2 + (y_j - y_{j-1})^2}} \delta_{i,j} - \frac{y_j - y_{j-1}}{\sqrt{\Delta x^2 + (y_j - y_{j-1})^2}} \delta_{i,j-1} \right]$$

$$\Rightarrow \left.\frac{\partial S}{\partial y_i}\right|_0 = \frac{y_i - y_{i-1}}{\sqrt{\Delta x^2 + (y_i - y_{i-1})^2}} - \frac{y_{i+1} - y_i}{\sqrt{\Delta x^2 + (y_{i+1} - y_i)^2}} = 0$$

solution $\Rightarrow$ $\quad y_i - y_{i-1} = y_{i+1} - y_i \quad \forall\, i = 1, \dots, N$

$$\therefore \quad y = \frac{h}{\ell} x \quad \forall\, (x_i, y_i)$$

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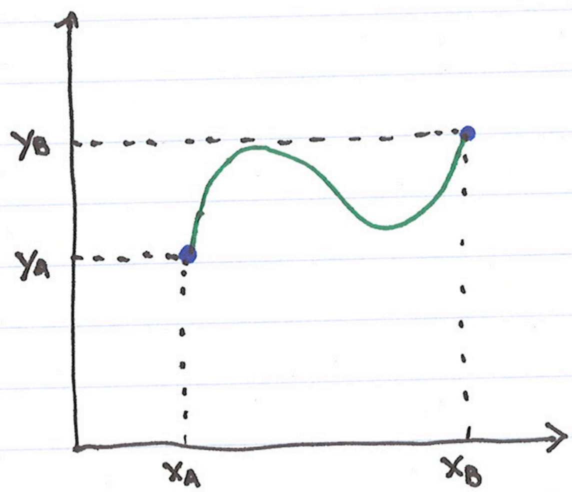

What curve $y(x)$ has the shortest path length S between the endpoints (x_A, y_A) and (x_B, y_B)?

How can we prove this?

Calculus of Variations

The total path length S is

$$S = \int_A^B ds = \int_{x_A}^{x_B} dx \sqrt{1+[y'(x)]^2}$$

since $\quad ds = \sqrt{dx^2+dy^2}$

End Points (x_A, y_A) and (x_B, y_B) are fixed

W.L.O.G we can set $\quad x_A = y_A = 0$

$x_B = \ell \quad y_B = h$

We can choose any path $y(x)$ we like and S depends on the entire function $y(x)$ (actually $y'(x)$ in this case). We should somehow denote this and we write

$$S = S[y(x)] = \int_0^\ell dx \sqrt{1+[y'(x)]^2}$$

S is a functional of $y(x)$

(of 1 function)