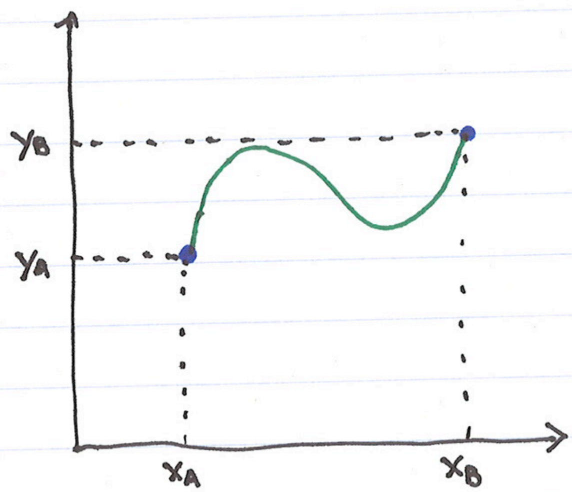
Calculus of Variations

The Shortest Path Between Two Points

Consider two points (x_A, y_A) and (x_B, y_B) in the (x,y) plane



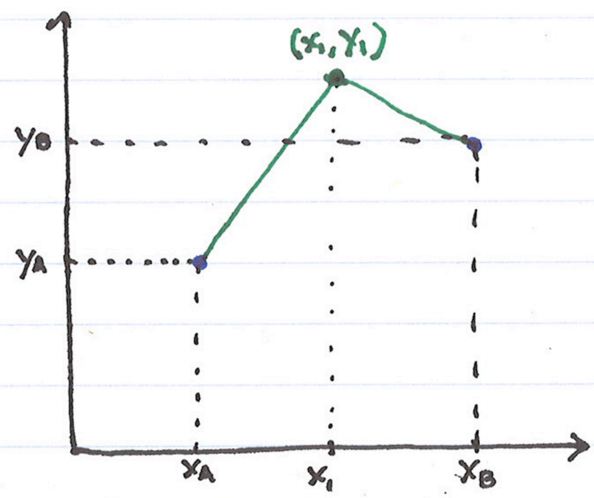
What curve $y(x)$ has the shortest path length S between the endpoints (x_A, y_A) and (x_B, y_B)?

How can we prove this?

Calculus of Variations

The Shortest 2-segment Path Between Two Points

Consider two points (x_A, y_A) and (x_B, y_B) in the (x, y) plane



What value of y_1 leads to the shortest 2-segment path length S between the end points (x_A, y_A) and (x_B, y_B)?

How can we prove this?

Calculus of Variations

The total path length S in the 2-segment case is

$$S = S_1 + S_2 = \sum_{j=1}^{2} S_j$$

where $S_1 = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

$$S_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or... $S_j = \sqrt{(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2}$

End Points $(x_0, y_0) = (x_A, y_A)$
$(x_2, y_2) = (x_B, y_B)$

W.L.O.G we can set $x_0 = y_0 = 0$
$x_2 = \ell \quad y_2 = h$

$$\Rightarrow \quad S = S(y_1) = \sqrt{\tfrac{1}{4}\ell^2 + y_1^2} + \sqrt{\tfrac{1}{4}\ell^2 + (h - y_1)^2}$$

Where we have taken $x_2 - x_1 = x_1 - x_0 = \frac{1}{2}\ell$

S is a function of y_1 (of 1 variable)

Calculus of Variations

Our goal is to minimize $S(y_1)$ at S_{min} for some value y_{1min}.

However, since up until now y_1 has been arbitrary we are free to choose $y_1 = y_{1min}$ (assuming a minumum exists) and paramatrize the variation away from y_1 with a parameter α.

That is, we can write $Y_1 = y_1 + \alpha$

so that $S = S(Y_1) = S(y_1 + \alpha)$

Taylor expanding about the minimum we have

$$S(Y_1) = \underbrace{S(y_1)}_{S_{min}} + \left.\frac{\partial S}{\partial \alpha}\right|_{\alpha=0} \alpha + \dots$$

A necessary condition for $S(y_1) = S_{min}$ is

$$\left.\frac{\partial S}{\partial \alpha}\right|_{0} = 0$$

Calculus of Variations

$$s(y_1) = \sqrt{\tfrac{1}{4}\ell^2 + (y_1+\alpha)^2} + \sqrt{\tfrac{1}{4}\ell^2 + (h-y_1-\alpha)^2}$$

$$\Rightarrow \quad \left.\frac{\partial s}{\partial \alpha}\right|_0 = \frac{y_1}{\sqrt{\tfrac{1}{4}\ell^2 + y_1^2}} - \frac{h-y_1}{\sqrt{\tfrac{1}{4}\ell^2 + (h-y_1)^2}} = 0$$

Algebraic Equation for y_1

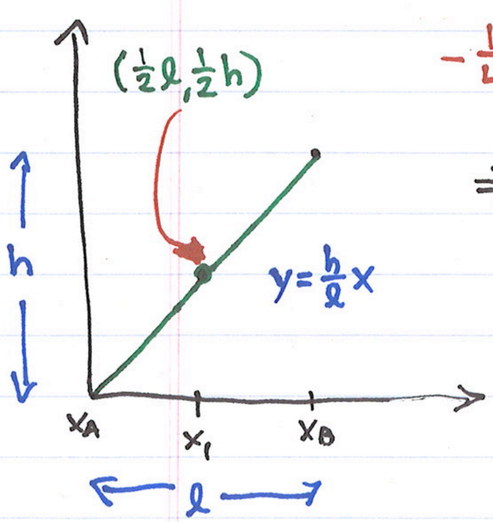
Solving for y_1:

$$y_1\sqrt{\tfrac{1}{4}\ell^2 + (h-y_1)^2} = (h-y_1)\sqrt{\tfrac{1}{4}\ell^2 + y_1^2}$$

$$y_1^2\left[\tfrac{1}{4}\ell^2 + (h-y_1)^2\right] = (h-y_1)^2\left[\tfrac{1}{4}\ell^2 + y_1^2\right]$$

$$\left[\tfrac{1}{4}\ell^2 + y_1^2 - \tfrac{1}{4}\ell^2\right]\left[\tfrac{1}{4}\ell^2 + (h-y_1)^2\right] = \left[\tfrac{1}{4}\ell^2 + y_1^2\right]\left[\tfrac{1}{4}\ell^2 + (h-y_1)^2 - \tfrac{1}{4}\ell^2\right]$$

$$-\tfrac{1}{4}\ell^2\left[\tfrac{1}{4}\ell^2 + (h-y_1)^2\right] = -\tfrac{1}{4}\ell^2\left[\tfrac{1}{4}\ell^2 + y_1^2\right]$$



$$\Rightarrow \quad (h-y_1)^2 = y_1^2$$

$$h^2 - 2hy_1 = 0$$

$$\therefore \quad y_1 = \tfrac{1}{2}h$$