Rotation of Ridged Bodies

Reminders... useful for ridged bodies

If we define the centre of mass coordinate

$$\vec{R} = \frac{1}{M}\int d^3\vec{r}\,\rho(\vec{r})\,\vec{r} \quad ; \quad M = \int d^3\vec{r}\,\rho(\vec{r})$$

Discrete case

total momentum $\vec{P} = M\dot{\vec{R}}$ $\longrightarrow$ $\vec{R} = \frac{1}{M}\sum_\alpha m_\alpha \vec{r}_\alpha$

Then we can show, with $\vec{r}\,' \equiv \vec{r} - \vec{R}$

total angular momentum $\vec{L} = \underbrace{\vec{R}\times\vec{P}}_{\text{motion of CoM}} + \underbrace{\sum_\alpha \vec{r}_\alpha\,' \times m_\alpha \vec{r}_\alpha\,'}_{\text{motion relative to CoM}}$

"orbital" "spin"

total Kinetic energy $T = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{\text{motion of CoM}} + \underbrace{\frac{1}{2}\sum_\alpha m_\alpha(\dot{\vec{r}}_\alpha\,')^2}_{\text{motion relative to CoM}}$

"bulk KE" "rotational KE"

total potential energy $U = U_{external} + U_{internal}$

constant for ridged body

To describe the rotational motion of a ridged body we want

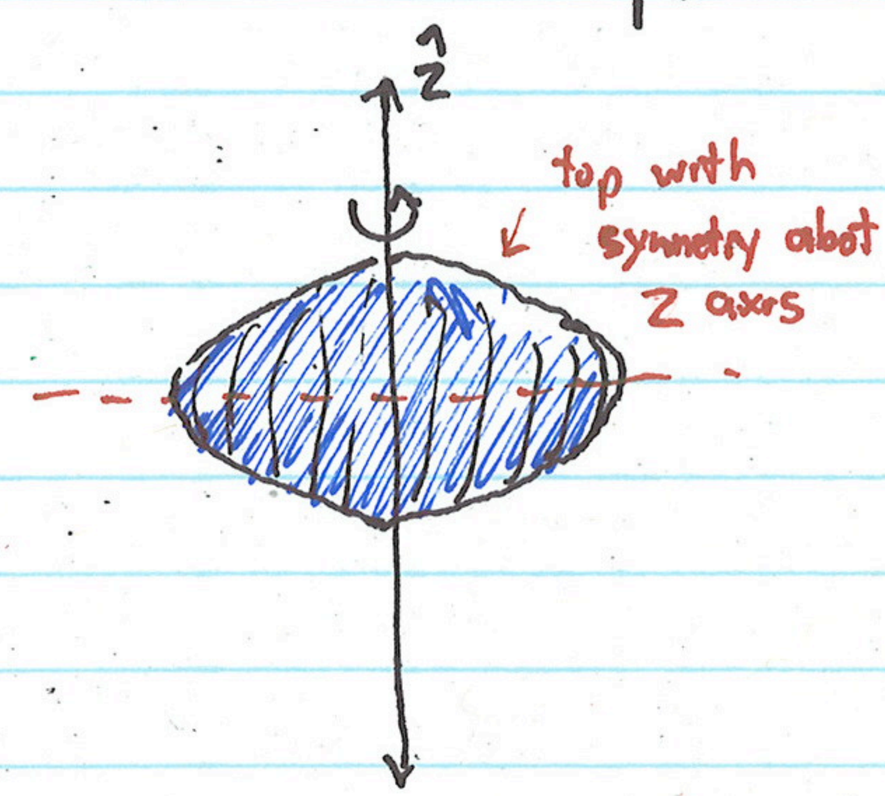
$$\boxed{\mathcal{L} = T_{rot} - U_{ext}}$$

Rotation of Ridged Bodies

When a Ridged Body (RB) is rotating about a symmetry axis we usually have the relationships

$T = \frac{1}{2} I \omega^2$ Kinetic Energy

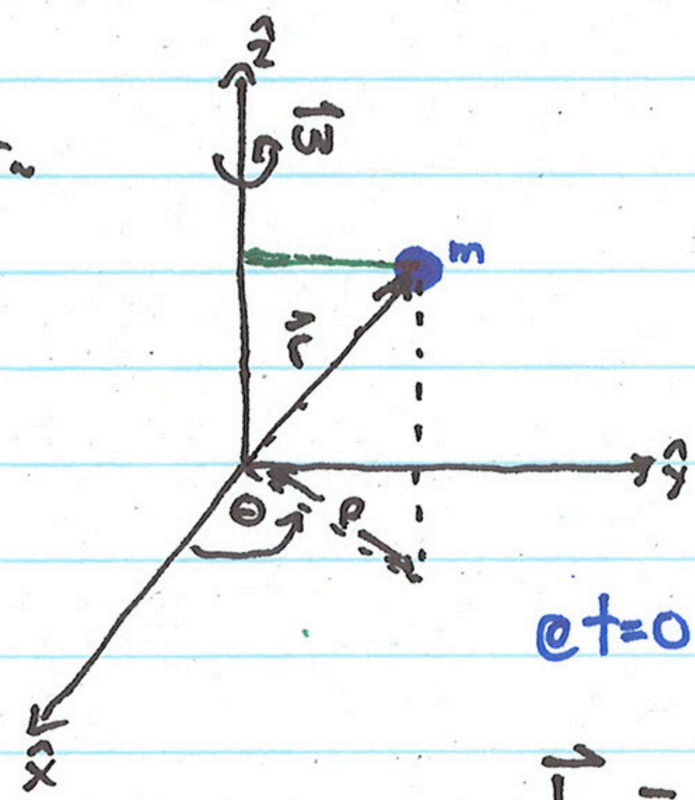
$\vec{L} = I \vec{\omega}$ Angular Momentum



where I is the moment of inertia about $\hat{\omega}$

However in general $\vec{L}$ and $\vec{\omega}$ point in different directions

Consider.



point mass rotating about $\hat{z}$

$$\vec{r} = \rho\cos\omega t\,\hat{i} + \rho\sin\omega t\,\hat{j} + z\hat{k}$$

$$\dot{\vec{r}} = -\omega\rho\sin\omega t\,\hat{i} + \omega\rho\cos\omega t\,\hat{j}$$

@t=0 $\dot{\vec{r}} = \omega\rho\hat{j}$ $\vec{r} = \rho\hat{i} + z\hat{k}$

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha$$

Now $\vec{\omega} = \omega\hat{k}$

$$= m(\omega\rho^2(\hat{i}\times\hat{j}) + \omega\rho z(\hat{k}\times\hat{j}))$$

$$\vec{L} = m\rho^2\omega\hat{k} - m\rho z\omega\hat{i}$$

$$\vec{L} = I_{zz}\omega\hat{k} + I_{xz}\omega\hat{i} \quad @t=0$$

Rotation of Ridged Bodies

In this case we know that the kinetic energy of the circular motion is just

$$T = \frac{1}{2} m|\vec{v}|^2 = \frac{1}{2} m(\rho\omega)^2 = \frac{1}{2}(m\rho^2)\omega^2$$

or in terms of $I_{zz} = m\rho^2$

$$T = \frac{1}{2} I_{zz}\omega^2 = \frac{1}{2}\vec{\omega}\cdot\vec{L}$$

since $\vec{\omega} = \omega\hat{k}$ $\quad \vec{L} = I_{zz}\omega\hat{k} + I_{xz}\omega\hat{i} + I_{yz}\omega\hat{j}$

It can be shown that generally the kinetic energy of rotation of a RB takes the form

$$\boxed{T = \frac{1}{2}\vec{\omega}\cdot\vec{L}}$$

What is the relationship between $\vec{L}$ and $\vec{\omega}$?

Rotation of Ridged Bodies

In general the vectors $\vec{L}$ and $\vec{\omega}$ are related linearly as

$$
\begin{aligned}
L_x &= I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z \\
L_y &= I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z \\
L_z &= I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z
\end{aligned}
$$

or with $1 \leftrightarrow x \quad 2 \leftrightarrow y \quad 3 \leftrightarrow z$

$$L_i = \sum_{j=1}^{3} I_{ij}\,\omega_j$$

or in Matrix form

$$
\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix}
=
\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}
\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}
$$

$$\vec{L} = \underline{\underline{I}}\,\vec{\omega}$$

Where $\underline{\underline{I}}$ is the Inertia Tensor for rotations about some point O
(may or may not be CoM)

Rotation of Ridged Bodies

For a body with mass density $\rho(\vec{r})$, and a particular choice of origin $\mathcal{O}$ then the components of $\underline{\underline{I}}$ are:

diagonal

$$I_{xx} = \int d^3\vec{r}\, \rho(\vec{r})(y^2+z^2)$$

$$I_{yy} = \int d^3\vec{r}\, \rho(\vec{r})(z^2+x^2)$$

$$I_{zz} = \int d^3\vec{r}\, \rho(\vec{r})(x^2+y^2)$$

off-diagonal

$$I_{xy} = I_{yx} = -\int d^3\vec{r}\, \rho(\vec{r})\, xy$$

$$I_{yz} = I_{zy} = -\int d^3\vec{r}\, \rho(\vec{r})\, yz$$

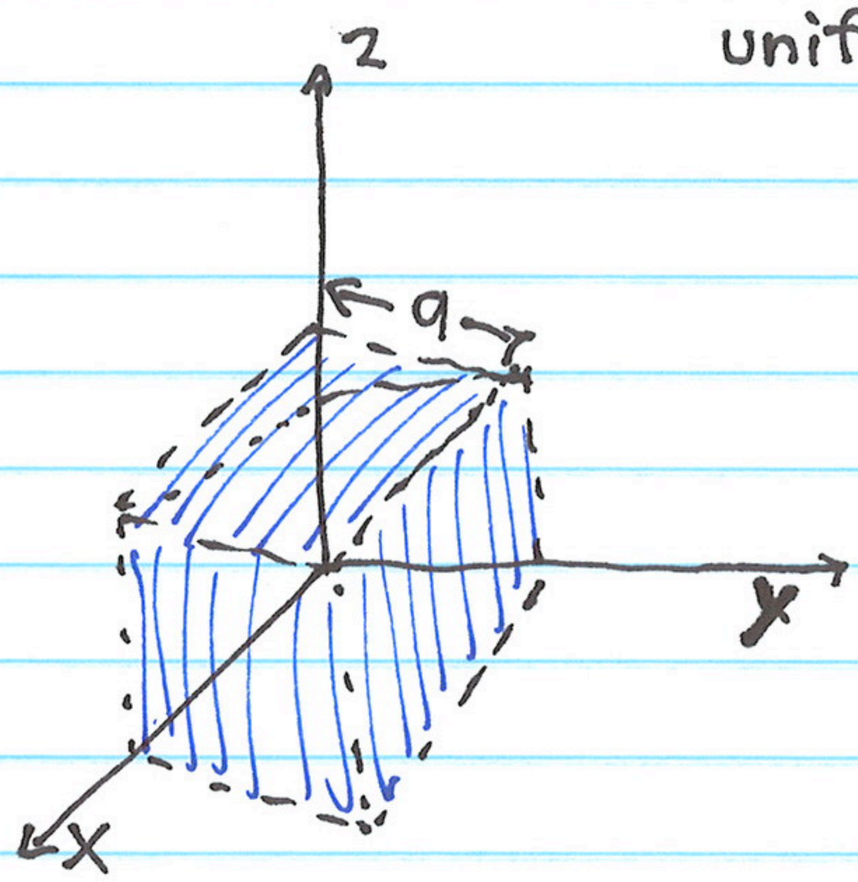
$$I_{zx} = I_{xz} = -\int d^3\vec{r}\, \rho(\vec{r})\, zx$$

For a discrete set of points $\int \to \sum$

Rotation of Ridged Bodies

Example: Cube rotating about corner.

uniform density ρ



What is $\underline{\underline{I}}$?

$$M = \rho a^3$$

Moment of Inertia of a Cube about corner.

$$I_{xx} = \iiint d^3\vec{r}\, \rho (y^2 + z^2)$$

$$\rho \int_0^a dx \int_0^a dy \int_0^a dz\, y^2 = \frac{1}{3}\rho a^5$$

$$\rho \left(\int_0^a dx\right)\left(\int_0^a dy\, y^2\right)\left(\int_0^a dz\right) = \rho a \left(\frac{1}{3}a^3\right) a$$

$$\rho \int_0^a dx \int_0^a dz \int_0^a dy\, z^2 = \frac{1}{3}\rho a^5$$

$$I_{yy} = I_{zz} = I_{xx} = \frac{2}{3}\rho a^5 = \frac{2}{3} M a^2$$

$$I_{xy} = -\iiint d^3\vec{r}\, \rho\, xy$$

$$= -\left(\int_0^a dx\, x\right)\left(\int_0^a dy\, y\right)\left(\int_0^a dz\right)\rho$$

$$I_{xy} = -\frac{1}{4}\rho a^5 = -\frac{1}{4} M a^2 = I_{yx}$$

$$\underline{\underline{I}} = \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix} M a^2$$