Rotation of Ridged Bodies

Centre of Mass Properties

coordinate $\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \quad \left(\vec{R} = \frac{1}{M}\int dm\, \vec{r}\right)$

total momentum $\vec{P} = \sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha$

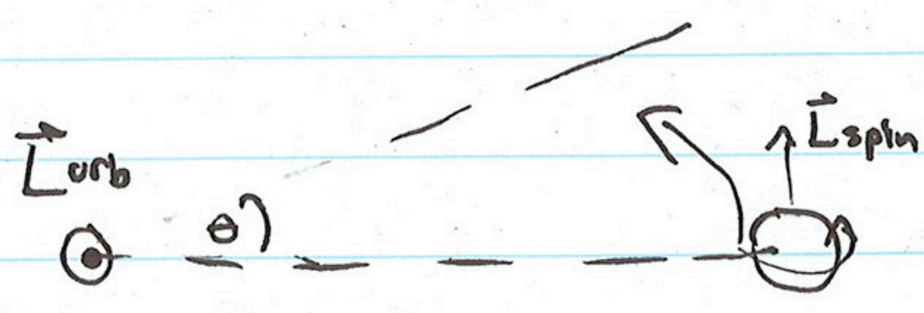
total angular momentum
$$\begin{cases} \vec{L} = \sum_\alpha \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha \\ \vec{L} = \vec{R}\times\vec{P} + \sum_\alpha \vec{r}_\alpha' \times m_\alpha \dot{\vec{r}}_\alpha' \end{cases}$$

where $\vec{r}_\alpha' \equiv \vec{r}_\alpha - \vec{R}$ is the coordinate relative to cofm

$$\vec{L} = \left(\begin{array}{c}\text{Angular Momentum}\\ \text{due to motion of cm}\end{array}\right) + \left(\begin{array}{c}\text{Angular Momentum}\\ \text{due to relative motion}\\ \text{about cofm}\end{array}\right)$$

ie. Example: Planet

$$\vec{L} = \vec{L}_{orb} + \vec{L}_{spin}$$



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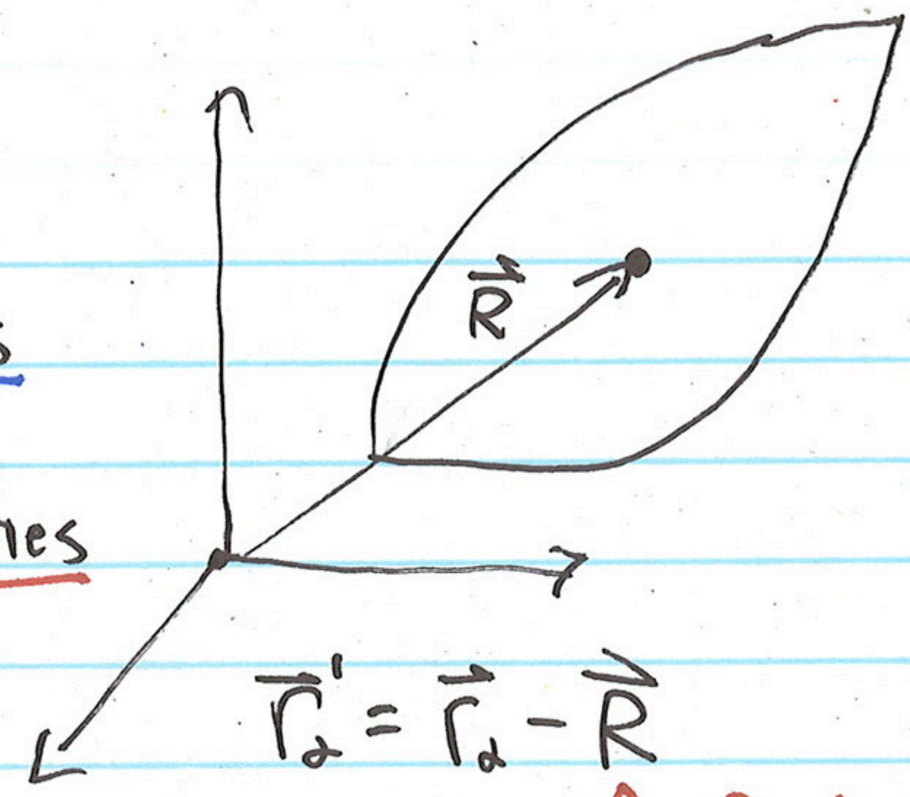


$$\vec{r}_\alpha' = \vec{r}_\alpha - \vec{R}$$

↑ fixed point in body

total Kinetic energy

$$\begin{cases} T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2} \\ T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum_\alpha m_\alpha (\dot{\vec{r}}_\alpha')^2 \\ T = T_{cm} + \underbrace{T_{rot}} \end{cases}$$

For a ridged body rotation is all we can have

Usually we take $\vec{R}$ to be the location of C of M but carry be any fixed point

ie. If no external forces $\vec{R} = 0$ choose c of m

total potential energy

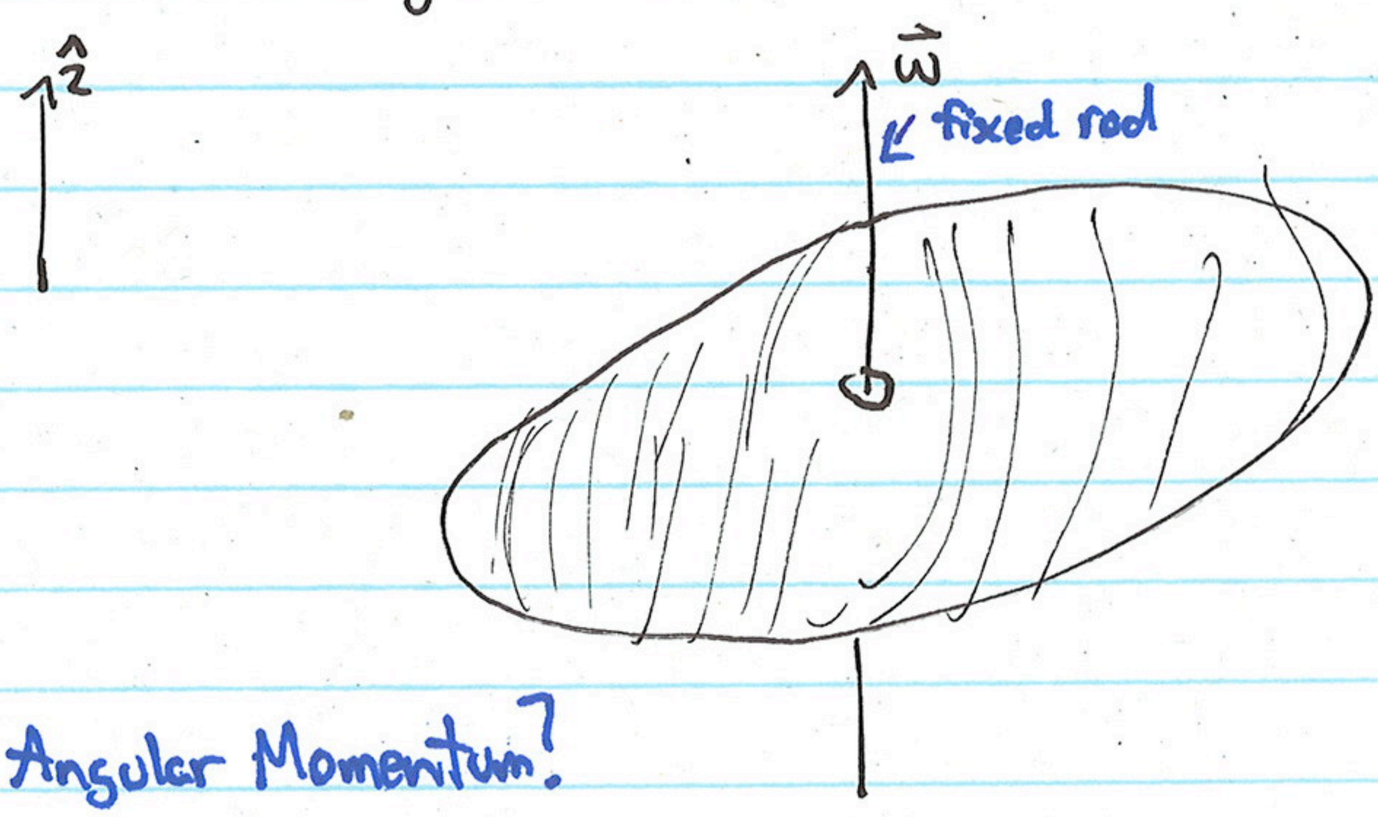
$$U = U_{ext} + \underbrace{U_{int}}_{\text{constant for ridged body}}$$

We only need to examine external forces when writing down U for a ridged body

Rotation of Ridged Bodies

Rotation about a Fixed Axis

Cosider a ridged body with a hole drilled through it:



What is Angular Momentum?

$$\vec{L} = \sum_\alpha \vec{l}_\alpha = \sum_\alpha \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$$

$$\vec{\omega} = (0, 0, \omega) = \omega \hat{z}$$

so $\quad \vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha = (-\omega y_\alpha, \omega x_\alpha, 0)$

and $\quad l_\alpha = m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = m_\alpha \omega (-z_\alpha x_\alpha, -z_\alpha y_\alpha, x_\alpha^2 + y_\alpha^2)$

Aside on Cross Products

$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j}$$

cyclic permutations

$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j}$$

$$\vec{w} = (0, 0, w) = w\hat{k}$$

$$\vec{v} = \vec{w} \times \vec{r} = (w\hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & w \\ x & y & z \end{vmatrix} = wy(\hat{k} \times \hat{j}) + wx(\hat{k} \times \hat{i})$$

$$\vec{v} = (-wy, -(-wx), \phi)$$

$$\vec{v} = (-wy, wx, 0) = -wy\hat{i} + wx\hat{j}$$

$$\vec{l} = m\,\vec{r} \times \vec{v} = m(x\hat{i} + y\hat{j} + z\hat{k}) \times (-wy\hat{i} + wx\hat{j})$$

$$m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ -wy & wx & 0 \end{vmatrix} = mw(x^2\,\hat{i} \times \hat{j} + y^2(-\hat{j} \times \hat{i}) + yz(-\hat{k} \times \hat{i}) + xz\,\hat{k} \times \hat{j})$$

$$\vec{l} = (-wxz, -(-wyz), wx^2 + wy^2)m$$

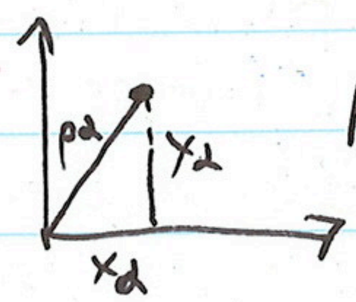
$$\vec{l} = mw(-xz, -yz, x^2 + y^2) = mw(-xz\hat{i} - yz\hat{j} + (x^2 + y^2)\hat{k})$$

Rotation of Ridged Bodies

We can assemble the angular momentum component by component.

$$L_z = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)\omega$$

if we called



$$\rho_\alpha^2 = x_\alpha^2 + y_\alpha^2$$

$$L_z = \sum_\alpha m_\alpha \rho_\alpha^2 \omega = I_z \omega$$

$I_z = \sum_\alpha m_\alpha \rho_\alpha^2$ is moment of inertia about the z axis

We can now calculate $T = \frac{1}{2} \sum_\alpha m_\alpha V_\alpha^2$

For each α $\quad V_\alpha = \rho_\alpha \omega$ (circular motion)

$$\Rightarrow \boxed{T = \frac{1}{2} \sum_\alpha m_\alpha \rho_\alpha^2 \omega^2 = \frac{1}{2} I_z \omega^2}$$