Kepler Problem

The radial equation is

Gravity

$$\mu \frac{d^2 r}{dt^2} = \mu \ddot{r} = -\frac{\partial U_{eff}}{\partial r} = -\frac{\gamma}{r^2} + \frac{\ell^2}{2\mu r^3}$$

Angular Momentum Conservation

sove for $r(t)$

We can use the fact that $\ell = r^2 \frac{d\phi}{dt}\mu$
to write $\qquad dt = \frac{r^2 \mu}{\ell} d\phi$

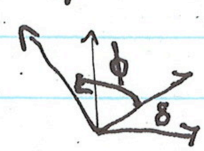
$$\Rightarrow \qquad \frac{d}{dt} = \frac{\ell}{r^2 \mu}\frac{d}{d\phi} = \frac{\ell}{\mu} u^2 \frac{d}{d\phi}$$

If we define $u = 1/r$ then the radial equation becomes

$$\mu \cdot \frac{\ell}{\mu} u^2 \frac{d}{d\phi}\left(\frac{\ell}{\mu} u^2 \frac{d}{d\phi}\left(1/u\right)\right) = -\gamma u^2 + \frac{\ell^2}{2\mu} u^3$$

$$\Rightarrow \qquad \frac{d^2 u}{d\phi^2} + u = \frac{\gamma \mu}{\ell^2}$$

"shifted Harmonic Oscillator Equation"



$$\Rightarrow \quad u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos(\phi + \delta)$$

We can always set $\delta = 0$ by redefining $\phi = 0$

Lagrangian Mechanics

Kepler Problem: The Orbit

$$\frac{d^2 u}{d\phi^2} + u(\phi) = \frac{\gamma\mu}{\ell^2}$$

$$\Rightarrow \frac{d^2}{d\phi^2}\left(u - \frac{\gamma\mu}{\ell^2}\right) + \left(u - \frac{\gamma\mu}{\ell^2}\right) = 0$$

$$\Rightarrow \quad u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos(\phi + \delta)$$

Can choose $\delta = 0$ by redefining $\phi = 0$

$$\therefore \quad u(\phi) = \frac{\gamma\mu}{\ell^2}\left(1 + \epsilon\cos\phi\right)$$

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}$$

$$c = \frac{\ell^2}{\gamma\mu}$$

$\epsilon < 1$	ellipse
$\epsilon = 1$	parabola
$\epsilon \geq 1$	hyperbola

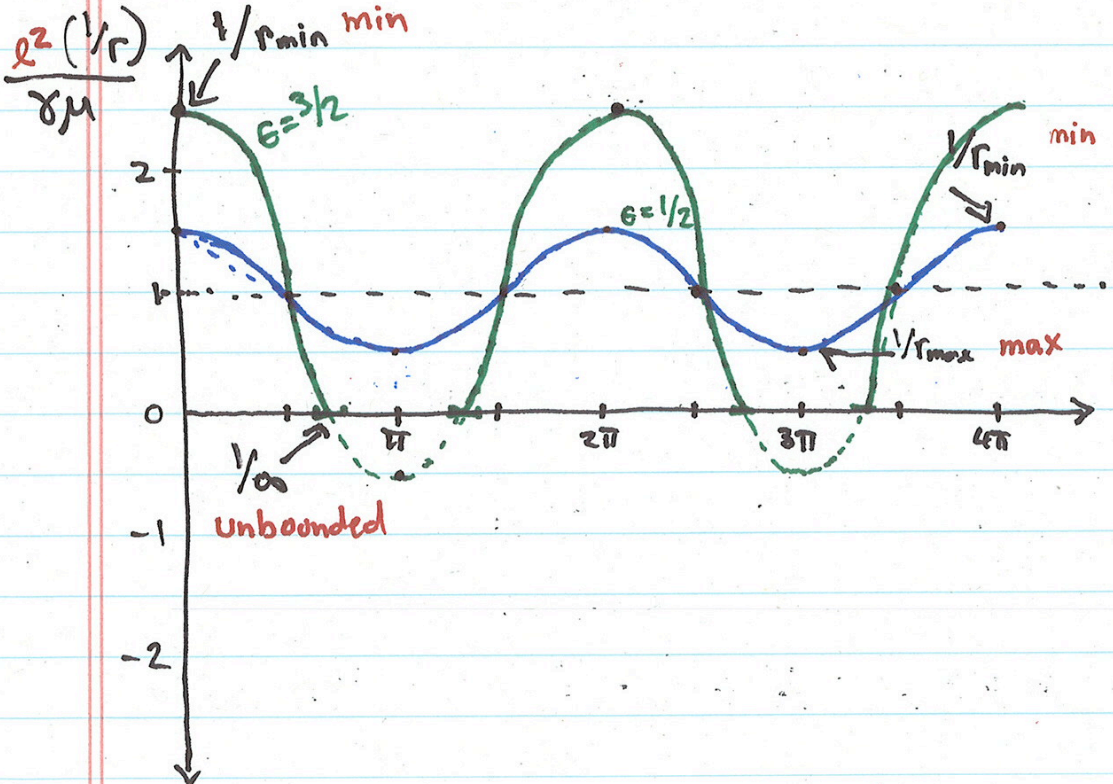
Kepler Problem

So $u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi$

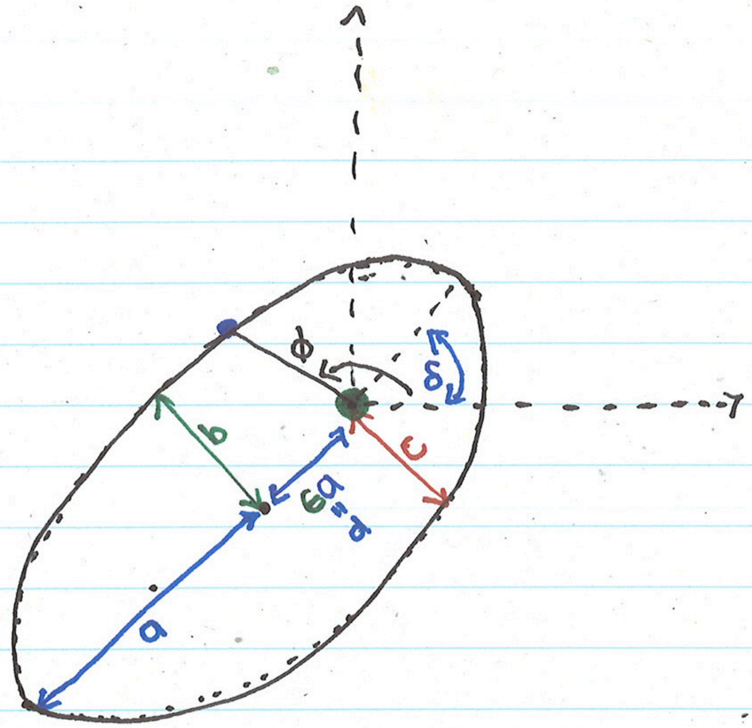
eccentricity

or $u(\phi) = \frac{\gamma\mu}{\ell^2}\left(1 + \epsilon\cos\phi\right)$

$\therefore \quad r(\phi) = \frac{(\ell^2/\gamma\mu)}{1+\epsilon\cos\phi} = \frac{c}{1+\epsilon\cos\phi}$



Kepler Problem



Orbital Parameters

$$U = -\frac{\gamma}{r}$$

$$r(\phi) = \frac{c}{1+\epsilon\cos(\phi-\delta)}$$

Hyperbola $\epsilon > 1$

$$\alpha = (\epsilon^2 - 1)\, c$$

Ellipse $\epsilon < 1$

$$\begin{cases} c = (1-\epsilon^2)\,a = \sqrt{1-\epsilon^2}\,b \\ b/a = \sqrt{1-\epsilon^2} \qquad d = \epsilon a \end{cases}$$

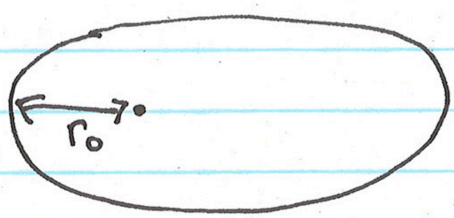
Angular Momentum $\quad c = \frac{\ell^2}{\gamma\mu} \qquad \ell = \sqrt{\gamma\mu c} = |\vec{r}\times\vec{p}|$

Energy $\quad E = -\frac{\gamma}{2a} = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1) = T + U$

$$= \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}$$

Kepler Problem

Example: Spaceship I



$\epsilon = \frac{1}{2}$

a) What is r_{max} (in terms of r_0)?
b) What are E and ℓ (in terms of r_0)?

---

c) What are the velocities at perigee (r_{min}) $= r_0$ and apogee (r_{max})?

impulse

d) Suppose the spaceship fires its rockets at perigee. What must its new velocity be in order to extend its orbit out to $5r_0 = r'_{max}$?

e) How could it end up on a circular orbit at $5r_0$?

a) $r_{max} = \frac{1+\epsilon}{1-\epsilon} r_0$

$\epsilon = \frac{1}{2} \qquad r_{max} = 3r_0$

b) $a = \frac{c}{1-\epsilon^2} \qquad \epsilon^2 = \frac{1}{4} \qquad a = \frac{4}{3}c$

$r_0 = \frac{c}{1+\epsilon} = \frac{2}{3}c \qquad a = 2r_0$

$\Rightarrow \quad E = -\frac{\gamma}{4r_0}$

$\ell = \sqrt{\gamma \mu c} = \sqrt{\frac{3}{2}\gamma \mu r_0}$

c) $\mu r_0 v_p = \ell = \mu (3r_0) v_a$

$\Rightarrow \quad v_p = \sqrt{\frac{3\gamma}{2\mu r_0}} \qquad v_a = \sqrt{\frac{\gamma}{6\mu r_0}}$

d) if $r_{max}' = 5r_0$ then $\frac{1+\epsilon'}{1-\epsilon'} = 5$

$1+\epsilon' = 5 - 5\epsilon'$

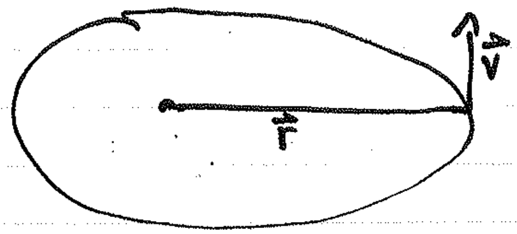
$\epsilon' = 2/3$

$\lambda = \sqrt{\frac{10}{9}}$

$v_f = \sqrt{\frac{10}{9}} v_0$

c) $\ell = |\vec{r} \times \vec{p}| = \mu |\vec{r} \times \vec{v}|$

$\ell = \mu r v$



$\ell = \mu r_0 v_p = \mu (3 r_0) v_a$

$\ell = \sqrt{\frac{3}{2} \gamma \mu r_0}$

$v_p = \sqrt{\frac{3\gamma}{2\mu r_0}}$ $\quad$ $v_a = \sqrt{\frac{\gamma}{6\mu r_0}}$

d) $\frac{c}{1+\epsilon} = \frac{c'}{1+\epsilon'}$

$r_{max} = \frac{1+\epsilon'}{1-\epsilon'} r_0 = 5 r_0$

$\epsilon' = 2/3$

$c \propto \ell^2$ $\quad$ $c' \propto (\ell')^2$ $\quad$ $\frac{c'}{c} = \lambda^2$

$\ell = \mu r_0 v_0$ $\quad$ $\ell' = \mu r_0 (\lambda v_0)$

d) cont. $\frac{c'}{c} = \lambda^2 = \frac{1+\epsilon'}{1+\epsilon}$ $\epsilon' = 2/3$

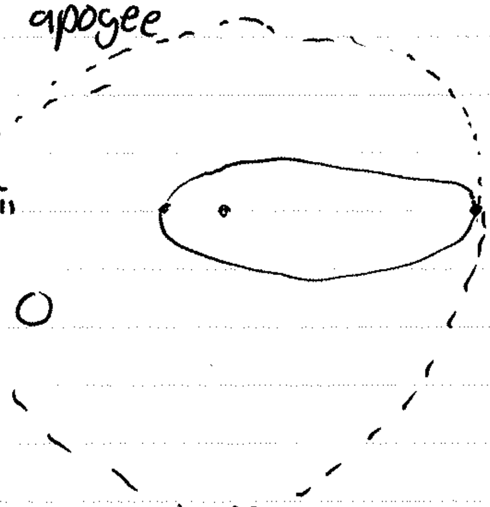
$\epsilon = 1/2$

$$= \frac{5/3}{3/2}$$

$$\lambda^2 = \frac{10}{9}$$

$$\lambda = \sqrt{\frac{10}{9}} \qquad V_0' = \sqrt{\frac{10}{9}}\, V_0$$

e) Fire the rocket at the apogee of the new orbit.



$$r_{max}' = 5r_0 = \frac{c'}{1-\epsilon'} = \frac{c''}{1-\epsilon''}$$

For a circular orbit $\epsilon'' = 0$

$$\frac{c''}{c'} = \lambda^2 = \frac{1}{1-\epsilon'} = \frac{1}{1-2/3} = 3$$

$V_0'' = \sqrt{3}\, V_0'$ $\lambda = \sqrt{3}$

Speed after impulse is increased by factor $\sqrt{3}$