Lagrangian Mechanics cont.

How to Solve Mechanics Problems

"The 6 Step Program"

1. Draw a picture to visualize the system.

2. Choose a set of generalized coordinates $\{q_1, q_2, \ldots, q_n\}$ that characterize the system.

3. Find the Kinetic Energy $T = T(q_i, \dot{q}_i, t)$ and the potential Energy $U = U(q_i, t)$ and the Lagrangian $L = T - U$. *Often it is easiest to write down T in terms of cartesian coordinates (of each particle) first.

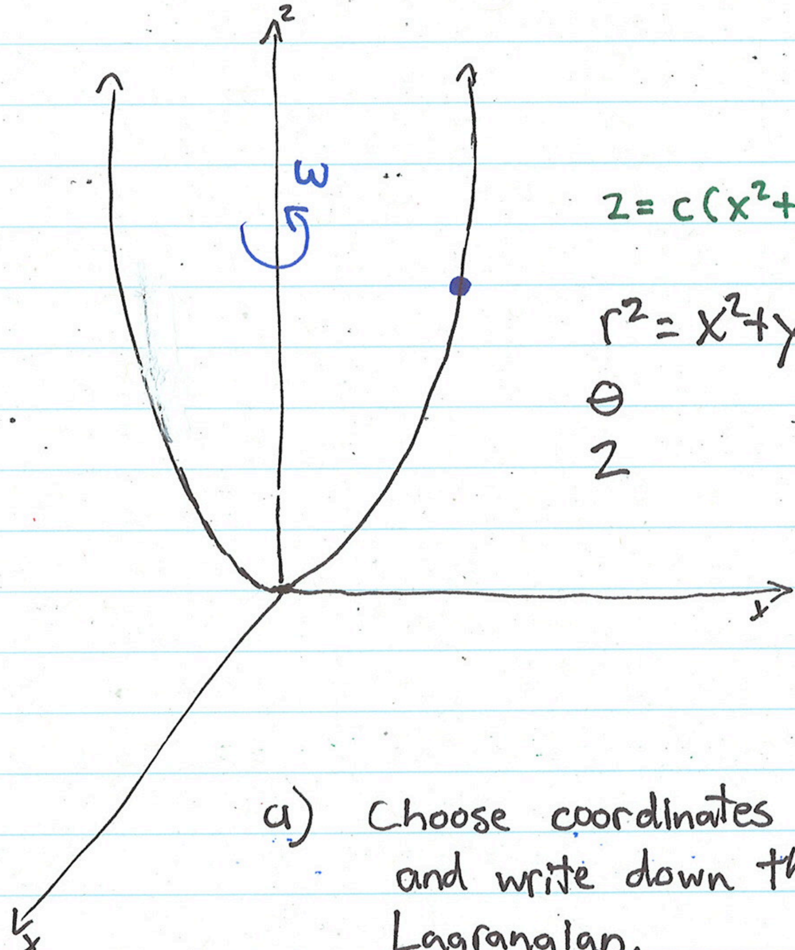
4. Find $\boxed{\frac{\partial L}{\partial \dot{q}_i} = \tilde{p}_i}$, generalized momenta and $\boxed{\frac{\partial L}{\partial q_i} = \tilde{F}_i}$, generalized forces.

5. Evaluate $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \dot{\tilde{p}}_i$ and write down the e.o.m $\boxed{\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}$ for each q_i. $\boxed{\tilde{F}_i = \dot{\tilde{p}}_i}$

6. Identify conserved quantities and solve diff. eqs.

Lagrangian Mechanics cont.

Example: Bead on a Rotating Parabola



$z = c(x^2+y^2)$

$r^2 = x^2+y^2$

θ

z

a) Choose coordinates and write down the Lagrangian.

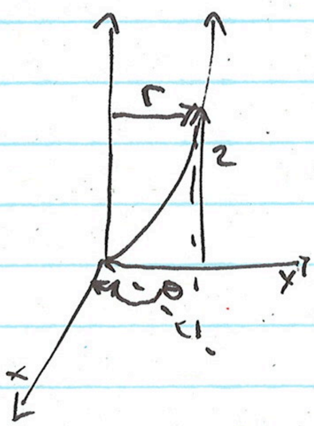
b) For what value of c will the bead remain/maintain constant height z for a given ω?

Lagrangian Mechanics cont.

Example: Bead on a Rotating Parabola

Constraints: $\dot{\theta} = \omega$

$z = c(x^2+y^2) = cr^2$



Choose r and θ

"trivial" $\theta = \omega t + \text{const}$

ignorable coordinate

Kinetic Energy $\quad T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

Now $\quad x = r\cos\theta \quad y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta + r\dot{\theta}(-\sin\theta) \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

$$\Rightarrow \dot{x}^2+\dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + r^2\omega^2$$

$$z = cr^2 \Rightarrow \dot{z} = 2c\dot{r}r$$

$$\Rightarrow T = \frac{1}{2}m(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2)$$

Lagrangian Mechanics cont

Potential Energy

$$U = mgz = mgcr^2$$

Lagrangian

$$L = \frac{1}{2}m\left(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2\right) - mgcr^2$$

$$\tilde{p}_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}(1+4c^2r^2)$$

$$\tilde{F}_r = \frac{\partial L}{\partial r} = m4c^2r\dot{r}^2 + mr\omega^2 - 2mgcr$$

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$m(4c^2r\dot{r}^2 + r\omega^2 - 2gcr) = m\ddot{r}(1+4c^2r^2) + m8c^2r\dot{r}$$

$$\therefore \ddot{r}(1+4c^2r^2) + \dot{r}^2(4c^2r) + r(2gc-\omega^2) = 0$$

if $\dot{r} = \ddot{r} = 0$

$$c = \frac{\omega^2}{2g}$$

```
c = 1
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1

```
g = 1 / 2
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$\frac{1}{2}$

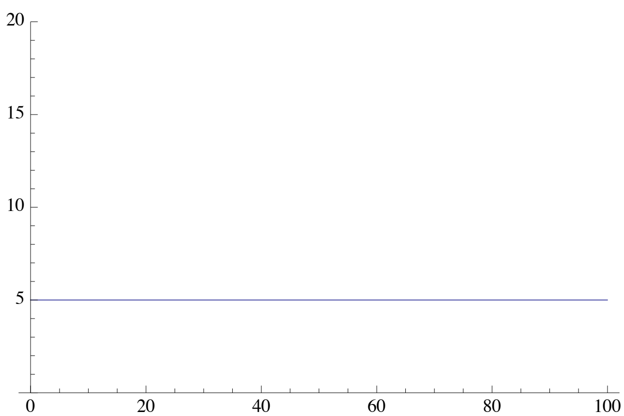
```
ω = 1
```

1

```
Solution = NDSolve[{r''[t] * (1 + 4 * c^2 * r[t]^2) + (4 * c^2 * (r'[t])) + r[t] * (2 * g * c - ω^2) == 0,
   r[0] == 5, r'[0] == 0}, r[t], {t, 0, 100}]
```

{{r[t] → InterpolatingFunction[{{0., 100.}}, <>][t]}}

```
Plot[{Evaluate[r[t]] /. Solution}, {t, 0, 100}, PlotRange → {0, 20}]
```




```
ω = 1.05
```

1.05

```
Solution = NDSolve[{r''[t] * (1 + 4 * c^2 * r[t]^2) + (4 * c^2 * (r'[t])) + r[t] * (2 * g * c - ω^2) == 0,
   r[0] == 5, r'[0] == 0}, r[t], {t, 0, 100}]
```

{{r[t] → InterpolatingFunction[{{0., 100.}}, <>][t]}}

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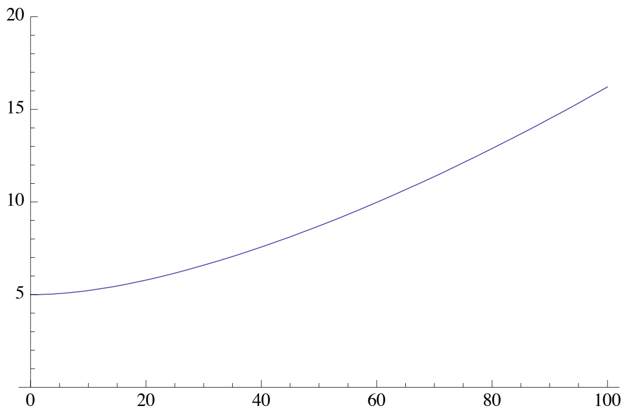
Plot[{Evaluate[r[t]] /. Solution}, {t, 0, 100}, PlotRange → {0, 20}]
```



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ω = 0.95
```

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0.95
```

```
Solution = NDSolve[{r''[t] * (1 + 4 * c^2 * r[t]^2) + (4 * c^2 * (r'[t])) + r[t] * (2 * g * c - ω^2) == 0,
   r[0] == 5, r'[0] == 0}, r[t], {t, 0, 100}]
```

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{{r[t] → InterpolatingFunction[{{0., 100.}}, <>][t]}}
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Plot[{Evaluate[r[t]] /. Solution}, {t, 0, 100}, PlotRange → {0, 20}]
```