Lagrangian Mechanics

Noether's Theorem

(Emmy Noether 1882-1935)

Every continuous symmetry of the Lagrangian is associated with a $\boxed{\text{conserved "charge"}}$

Consider $L = L(q_i, \dot{q}_i, t)$ $\quad q_i = \{q_1, q_2, \ldots, q_n\}$

and imagine a one-parameter family of transformations

$$q_i \to \bar{q}_i(q_i, \xi)$$

ie $\quad (x, y, z) \to (x+\xi, y, z) = (\bar{x}, \bar{y}, \bar{z})$

If the Lagrangian is invariant under this transformation then

$$L(q_i, \dot{q}_i, t) = L(\bar{q}_i, \dot{\bar{q}}_i, t)$$

ie. $\quad L(x, \dot{x}, t) = L(\bar{x}, \dot{\bar{x}}, t)$

Lagrangian Mechanics

Example: Momentum Conservation

Consider the transformation

$$(x, y, z) \to (\bar{x}, \bar{y}, \bar{z}) = (x + \xi, y, z)$$

$$\Rightarrow \frac{\partial \bar{x}}{\partial \xi} = 1 \qquad \frac{\partial \bar{y}}{\partial \xi} = 0 = \frac{\partial \bar{z}}{\partial \xi}$$

$$\Lambda = \frac{\partial L}{\partial \dot{x}} \frac{\partial \bar{x}}{\partial \xi} + \frac{\partial L}{\partial \dot{y}} \cancelto{0}{\frac{\partial \bar{y}}{\partial \xi}} + \cancelto{0}{\frac{\partial \bar{z}}{\partial \xi}} \frac{\partial L}{\partial \dot{z}}$$

$$\therefore \quad \Lambda = \frac{\partial L}{\partial \dot{x}} = p_x \quad \text{is conserved}$$

(provided $L(\bar{x}, \dot{\bar{x}}, t) = L(x, \dot{x}, t)$)

Lagrangian Mechanics

The Hamiltonian

Suppose the Lagrangian is invariant under time translations

$$t \rightarrow \tilde{t} + \xi$$

This means $$L(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t+\xi)$$

or... $$\frac{\partial L}{\partial \xi} = 0 = \frac{\partial L}{\partial t}$$

Now generally...

Explicit time dependence

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial L}{\partial t}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

but $$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{d}{dt}(p_i) = \dot{p}_i$$

$$\Rightarrow \quad \frac{dL}{dt} = \sum_i \left(\dot{p}_i \dot{q}_i + p_i \ddot{q}_i \right) + \frac{\partial L}{\partial t}$$

$$\therefore \quad \frac{dL}{dt} = \frac{d}{dt}\left(\sum_i p_i \dot{q}_i \right) + \frac{\partial L}{\partial t}$$

Lagrangian Mechanics

The Hamiltonian

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_i p_i \dot{q}_i\right) + \frac{\partial L}{\partial t}$$

let $$H = \sum_i p_i \dot{q}_i - L$$ "The Hamiltonian"

Then $$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

if we have time translation invariance
$$\frac{\partial L}{\partial t} = 0$$

$\Rightarrow$ H is conserved, is a constant of motion, if L does not explicitly depednd on time.

Lagrangian Mechanics cont.

What is $\sum_i p_i \dot{q}_i$?

Suppose $\vec{r}_\alpha = \vec{r}_\alpha(q_i)$ $\quad q_i = \{q_1, q_2, \ldots, q_n\}$

Then $\dot{\vec{r}}_\alpha = \sum_i \frac{\partial \vec{r}_\alpha}{\partial q_i} \dot{q}_i$

$$\dot{\vec{r}}_\alpha^{\,2} = \sum_j \sum_k \frac{\partial \vec{r}_\alpha}{\partial q_j} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_k} \dot{q}_j \dot{q}_k$$

So $T = \frac{1}{2} \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2} \sum_{j,k} A_{jk} \dot{q}_j \dot{q}_k$

$$A_{jk} \equiv \sum_\alpha m_\alpha \left(\frac{\partial \vec{r}_\alpha}{\partial q_j} \cdot \frac{\partial \vec{r}_\alpha}{\partial q_k} \right)$$

Now $p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_{j,k} A_{kj} \dot{q}_j \delta_{ik}$

$$p_i = \sum_j A_{ij} \dot{q}_j$$

$$\Rightarrow \boxed{\sum_i p_i \dot{q}_i = \sum_{i,j} A_{ij} \dot{q}_i \dot{q}_j = 2T}$$

$$\therefore \quad H = 2T - (T - U) = T + U$$

The Hamiltonian Function Is Equal to the Energy. Time Translation Invariance leads to energy conservation.

Lagrangian Mechanics cont.

How to Solve Mechanics Problems

"The 6 Step Program"

1. Draw a picture to visualize the system.

2. Choose a set of generalized coordinates $\{q_1, q_2, \ldots, q_n\}$ that characterize the system.

3. Find the Kinetic Energy $T = T(q_i, \dot{q}_i, t)$ and the potential Energy $U = U(q_i, t)$ and the Lagrangian $L = T - U$. *Often it is easiest to write down T in terms of cartesian coordinates (of each particle) first.

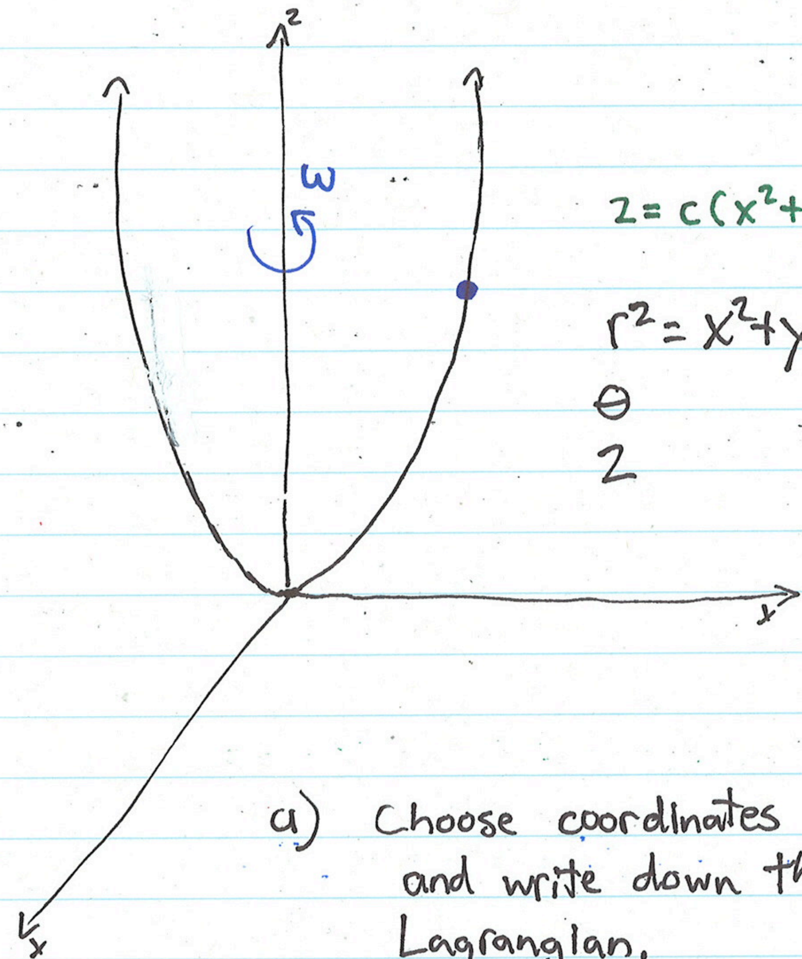
4. Find $\boxed{\frac{\partial L}{\partial \dot{q}_i} = \tilde{p}_i}$, generalized momenta and $\boxed{\frac{\partial L}{\partial q_i} = \tilde{F}_i}$, generalized forces.

5. Evaluate $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \dot{\tilde{p}}_i$ and write down the e.o.m $\boxed{\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}$ for each q_i. $\boxed{\tilde{F}_i = \dot{\tilde{p}}_i}$

6. Identify conserved quantities and solve diff. eqs.

Lagrangian Mechanics cont.

Example: Bead on a Rotating Parabola



$z = c(x^2+y^2)$

$r^2 = x^2+y^2$

θ

z

a) Choose coordinates and write down the Lagrangian.

b) For what value of c will the bead remain/maintain constant height z for a given ω?

Lagrangian Mechanics cont.

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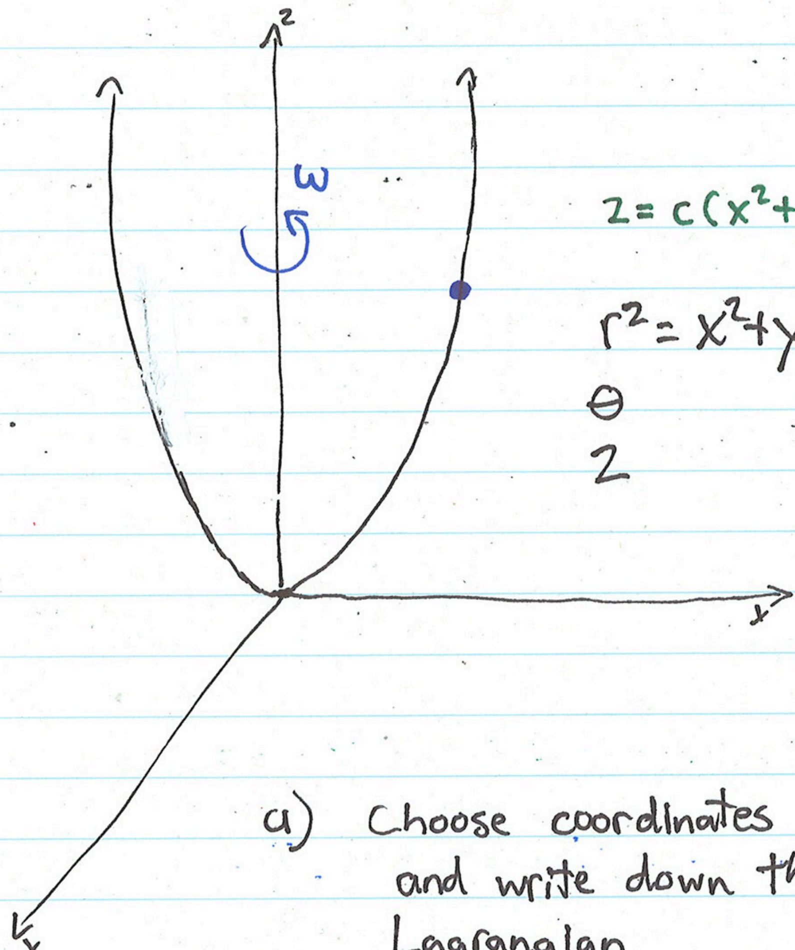

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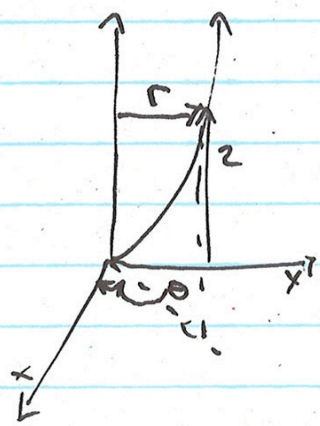
Lagrangian Mechanics cont

Example: Bead on a Rotating Parabola

Constraints: $\dot{\theta} = \omega$

$z = c(x^2+y^2) = cr^2$



Choose r and θ

"trivial" $\theta = \omega t + \text{const}$

ignorable coordinate

Kinetic Energy $\quad T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

Now $\quad x = r\cos\theta \quad y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta + r\dot{\theta}(-\sin\theta) \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

$$\Rightarrow \dot{x}^2+\dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + r^2\omega^2$$

$$z = cr^2 \Rightarrow \dot{z} = 2c\dot{r}r$$

$$\Rightarrow T = \frac{1}{2}m(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2)$$

Lagrangian Mechanics cont

Potential Energy

$$U = mgz = mgcr^2$$

Lagrangian

$$L = \frac{1}{2}m\left(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2\right) - mgcr^2$$

$$\tilde{p}_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}(1+4c^2r^2)$$

$$\tilde{F}_r = \frac{\partial L}{\partial r} = m4c^2r\dot{r}^2 + mr\omega^2 - 2mgcr$$

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$$

$$m(4c^2r\dot{r}^2 + r\omega^2 - 2gcr) = m\ddot{r}(1+4c^2r^2) + m8c^2r\dot{r}$$

$$\therefore \ddot{r}(1+4c^2r^2) + \dot{r}^2(4c^2r) + r(2gc-\omega^2) = 0$$

if $\dot{r} = \ddot{r} = 0$

$$c = \frac{\omega^2}{2g}$$

```
c = 1
```

```
1
```

```
g = 1 / 2
```

$\frac{1}{2}$

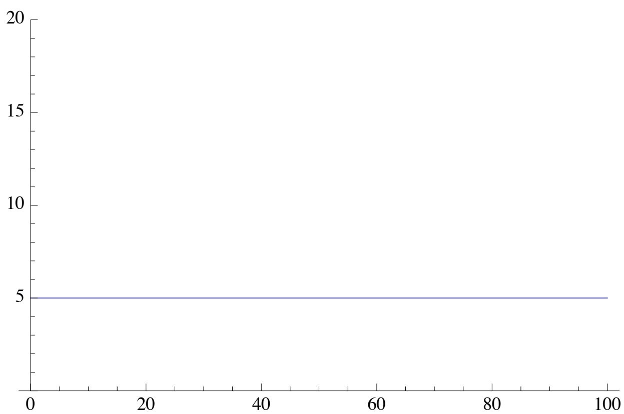
```
ω = 1
```

```
1
```

```
Solution = NDSolve[{r''[t] * (1 + 4 * c^2 * r[t]^2) + (4 * c^2 * (r'[t])) + r[t] * (2 * g * c - ω^2) == 0,
   r[0] == 5, r'[0] == 0}, r[t], {t, 0, 100}]
```

```
{{r[t] → InterpolatingFunction[{{0., 100.}}, <>][t]}}
```

```
Plot[{Evaluate[r[t]] /. Solution}, {t, 0, 100}, PlotRange → {0, 20}]
```




```
ω = 1.05
```

```
1.05
```

```
Solution = NDSolve[{r''[t] * (1 + 4 * c^2 * r[t]^2) + (4 * c^2 * (r'[t])) + r[t] * (2 * g * c - ω^2) == 0,
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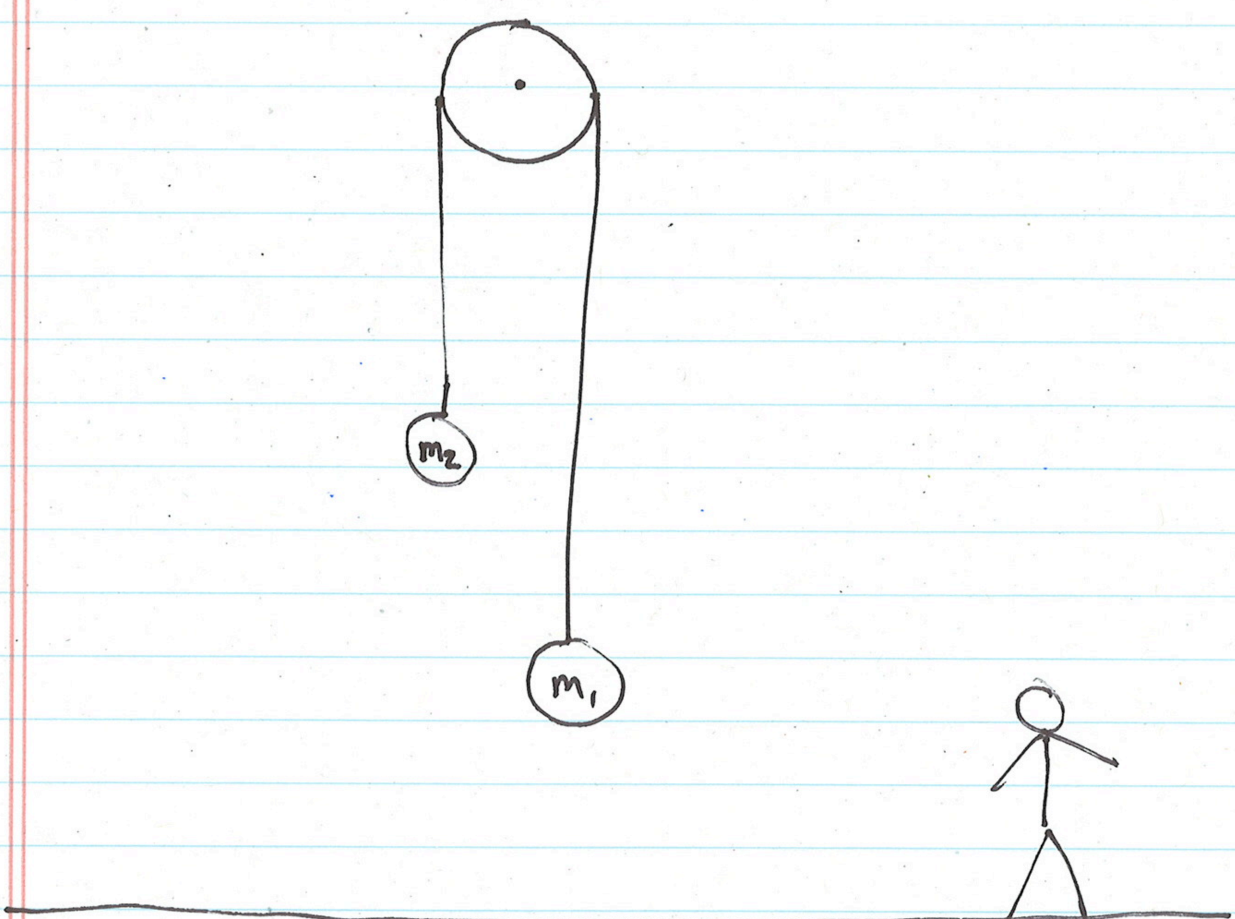
Lagrangian Mechanics cont.

Example: Atwood's Machine



a) Choose generalized coordinates and write down the Lagrangian.

b) Write down E.o.M.