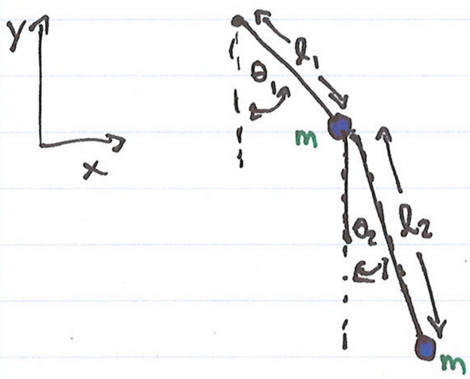
Lagrangian Mechanics cont.

Example: Double Pendulum



We can completely characterize this system with the angles θ_1 and θ_2

Kinetic Energy $\quad T = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2)$

$$x_1 = l_1 \sin\theta_1 \qquad \dot{x}_1 = +l_1 \cos\theta_1 \, \dot{\theta}_1$$
$$y_1 = -l_1 \cos\theta_1 \qquad \dot{y}_1 = +l_1 \sin\theta_1 \, \dot{\theta}_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \qquad \dot{x}_2 = \dot{x}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2$$
$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \qquad \dot{y}_2 = \dot{y}_1 + l_2 \sin\theta_2 \, \dot{\theta}_2$$

$$\Rightarrow \dot{x}_2^2 = \dot{x}_1^2 + 2 l_1 l_2 \cos\theta_1 \overset{\times \dot{\theta}_1 \dot{\theta}_2}{\cos\theta_2} + l_2^2 \cos^2\theta_2 \, \dot{\theta}_2^2$$
$$\dot{y}_2^2 = \dot{y}_1^2 + 2 l_1 l_2 \sin\theta_1 \underset{\times \dot{\theta}_1 \dot{\theta}_2}{\sin\theta_2} + l_2^2 \sin^2\theta_2 \, \dot{\theta}_2^2$$

$$\Rightarrow T = m l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2 + m l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2$$

Lagrangian Mechanics cont.

Example: The Double Pendulum cont.

Potential Energy $\quad U = mgy_1 + mgy_2$

$$\Rightarrow \quad U = -2mgl_1\cos\theta_1 - mgl_2\cos\theta_2$$

so

$$L = ml_1^2\dot{\theta}_1^2 + \frac{1}{2}ml_2^2\dot{\theta}_2^2 + ml_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + 2mgl_1\cos\theta_1 + mgl_2\cos\theta_2$$

Check $l_2 \to 0$ gives simple pendulum of mass $2m$

θ_1:

$$\frac{\partial L}{\partial \theta_1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)$$

$$\tilde{F}_1 = \frac{\partial L}{\partial \theta_1} = -ml_1l_2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 - 2mgl\sin\theta_1$$

$$\tilde{p}_1 = \frac{\partial L}{\partial \dot{\theta}_1} = 2ml_1^2\dot{\theta}_1 + ml_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_2$$

$$\Rightarrow \quad \ddot{\theta}_1 + \frac{l_2}{2l_1}\cos(\theta_1-\theta_2)\ddot{\theta}_2 + \frac{l_2}{2l_1}\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + \frac{g}{l_1}\sin\theta_1 = 0$$

Lagrangian Mechanics cont.

Example: The Double Pendulum cont.

θ_2: $$\frac{\partial L}{\partial \theta_2} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)$$

$$\tilde{F}_2 = \frac{\partial L}{\partial \theta_2} = + m l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2 - m g l_2 \sin\theta_2$$

$$\tilde{p}_2 = \frac{\partial L}{\partial \dot{\theta}_2} = m l_2^2 \dot{\theta}_2 + m l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1$$

$$\Rightarrow \quad \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + \frac{l_2}{l_1}\ddot{\theta}_2 - \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 + \frac{g}{l_2}\sin\theta_2 = 0$$

Equations of Motion:

$$\ddot{\theta}_1 + \frac{l_2}{2 l_1}\cos(\theta_1 - \theta_2)\ddot{\theta}_2 + \frac{l_2}{2 l_1}\sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + \frac{g}{l_1}\sin(\theta_1) = 0$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}\cos(\theta_1 - \theta_2)\ddot{\theta}_1 - \frac{l_1}{l_2}\sin(\theta_1 - \theta_2)\dot{\theta}_1^2 + \frac{g}{l_2}\sin(\theta_2) = 0$$

coupled nonlinear second order ODE's

Lagrangian Mechanics cont.

Example: The Double Pendulum cont.

Check: $\theta_1 = \theta_2 = 0$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 1 \qquad \sin(\theta_1) = \sin(\theta_2)$
$= \sin(\theta_1 - \theta_2) = 0$

$$\Rightarrow \quad \ddot{\theta}_1 + \frac{l_2}{2l_1}\ddot{\theta}_2 = 0 = \ddot{\theta}_2 + \frac{l_1}{l_2}\ddot{\theta}_1$$

$$\Rightarrow \quad \ddot{\theta}_1 = \ddot{\theta}_2 = 0$$

Equilibrium Point @ $\theta_1 = \theta_2 = 0$

Small Oscillations about Equilibrium

Near Equilibrium $\theta_1 \ll 1 \quad \theta_2 \ll 1$

To linear order
$$\cos(\theta_1 - \theta_2) \simeq 1 + o(\theta_1 - \theta_2)^2$$
$$\sin(\theta_1 - \theta_2) \simeq \theta_1 - \theta_2 + \cdots$$
$$\sin(\theta_1) \simeq \theta_1 + \cdots$$
$$\sin(\theta_2) \simeq \theta_2 + \cdots$$

Lagrangian Mechanics cont.

Example: The Double Pendulum cont.

substituting into the E.O.M

$$\ddot{\theta}_1 + \frac{l_2}{2l_1}(1)\ddot{\theta}_2 + \frac{l_2}{2l_1}(\theta_1 - \theta_2)\dot{\theta}_2^{\,2} + \frac{g}{l_1}(\theta_1) = 0$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}(1)\ddot{\theta}_1 - \frac{l_1}{l_2}(\theta_1 - \theta_2)\dot{\theta}_1^{\,2} + \frac{g}{l_2}(\theta_2) = 0$$

$$\therefore \quad \ddot{\theta}_1 + \frac{l_2}{2l_1}\ddot{\theta}_2 + \frac{g}{l_1}\theta_1 = 0$$

$$\ddot{\theta}_1 + \frac{l_2}{l_1}\ddot{\theta}_2 + \frac{g}{l_1}\theta_2 = 0$$

let $\gamma \equiv l_2/l_1 \quad \omega_0^2 \equiv g/l_1$

$$\Rightarrow \quad \ddot{\theta}_1 + \tfrac{1}{2}\gamma\ddot{\theta}_2 + \omega_0^2\theta_1 = 0 \qquad \text{Eq.1}$$
$$\ddot{\theta}_1 + \gamma\ddot{\theta}_2 + \omega_0^2\theta_2 = 0 \qquad \text{Eq.2}$$

"Trick" to diagonalize the system ~

$$(\text{Eq.1}) + C(\text{Eq.2}) = 0$$

Lagrangian Mechanics cont.

Example: The Double Pendulum cont.

$$\Rightarrow \quad (1+c)\ddot{\theta}_1 + \left(\tfrac{1}{2}+c\right)\gamma\ddot{\theta}_2 + \omega_0^2(\theta_1 + c\,\theta_2) = 0$$

Require: $$\frac{(\frac{1}{2}+c)\gamma}{1+c} = \frac{c\,\cancel{\omega_0^2}}{\cancel{\omega_0^2}}$$

$$\Rightarrow \quad c^2 + (1-\gamma)c - \tfrac{1}{2}\gamma = 0$$

$$c_\pm = \tfrac{1}{2}\left[(\gamma-1) \pm \sqrt{(\gamma-1)^2 + 2\gamma}\right]$$

$$\Rightarrow \quad c_\pm = \tfrac{1}{2}\left[(\gamma-1) \pm \sqrt{\gamma^2+1}\right]$$

If we define $\theta_\pm = \theta_1 + c_\pm \theta_2$

Then the E.O.M become

$$\ddot{\theta}_1 + c_\pm \ddot{\theta}_2 + \frac{\omega_0^2}{1+c_\pm}(\theta_1 + c_\pm \theta_2)$$

Normal Modes (decoupled equations) $$\ddot{\theta}_\pm + \omega_\pm^2 \theta_\pm = 0$$

Normal Frequencies $$\omega_\pm^2 = \frac{\omega_0^2}{1+c_\pm}$$

Lagrangian Mechanics cont.

Example: The Double Pendulum cont.

For $\gamma = 1$ $\qquad$ $\omega_0 = 1$

$$C_\pm = \pm \frac{1}{\sqrt{2}}$$

Θ_- mode: $\quad$ set $\Theta_+ = 0 = \Theta_1 + C_+ \Theta_2$

$$\Rightarrow \Theta_1 = -\frac{1}{\sqrt{2}} \Theta_2$$

$$\omega_- = \frac{1}{\sqrt{1 - \frac{1}{\sqrt{2}}}} = 1.848$$

Θ_+ mode: $\quad$ set $\Theta_- = 0 = \Theta_1 + C_- \Theta_2$

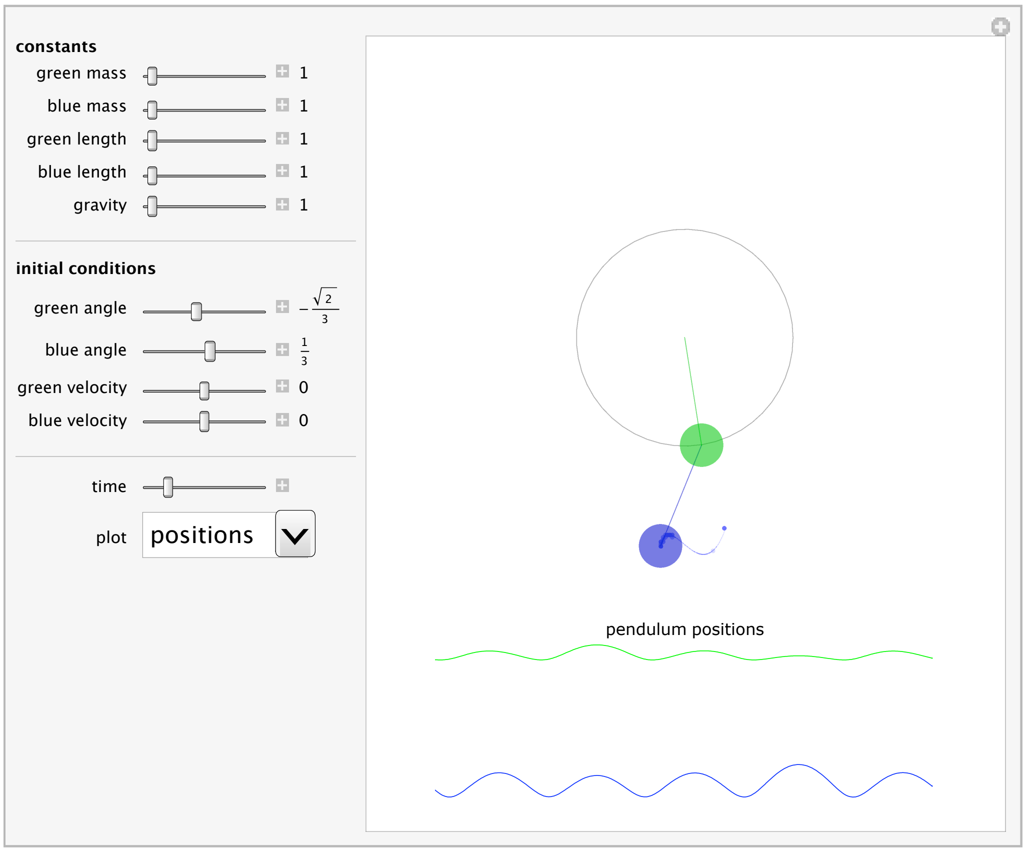
$$\Rightarrow \Theta_1 = \frac{1}{\sqrt{2}} \Theta_2$$

$$\omega_+ = \frac{1}{\sqrt{1 + \frac{1}{\sqrt{2}}}} = 0.7654$$

$$\frac{1.848}{0.7654} \simeq 1 + \sqrt{2} = 2.414$$

Wolfram Demonstrations Project

Double Pendulum



This shows the time evolution of a frictionless two-pendulum system. It is one of the simplest constructions to exhibit chaotic behavior.

THINGS TO TRY

Resize Images
- Slider Zoom

RELATED LINKS

- Double Pendulum (*ScienceWorld*)

Lagrangian Mechanics cont.

How to Solve Mechanics Problems

"The 6 Step Program"

1. Draw a picture to visualize the system.

2. Choose a set of generalized coordinates $\{q_1, q_2, \ldots, q_n\}$ that characterize the system.

3. Find the Kinetic Energy $T = T(q_i, \dot{q}_i, t)$ and the potential Energy $U = U(q_i, t)$ and the Lagrangian $L = T - U$. *Often it is easiest to write down T in terms of cartesian coordinates (of each particle) first.

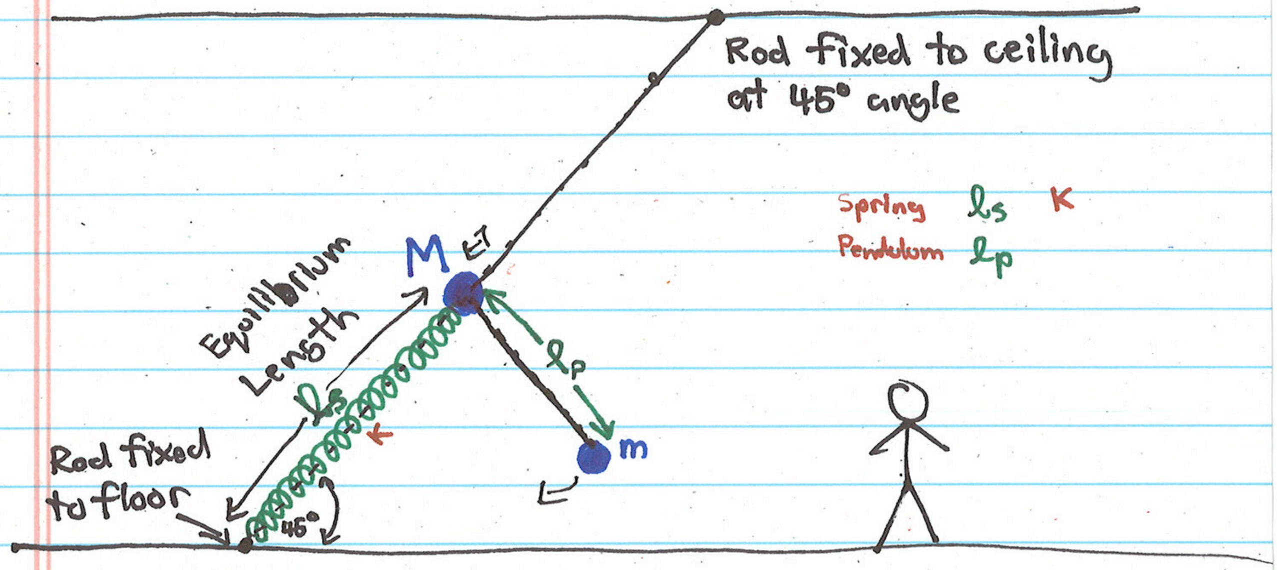
4. Find $\boxed{\frac{\partial L}{\partial \dot{q}_i} = \tilde{p}_i}$, generalized momenta and $\boxed{\frac{\partial L}{\partial q_i} = \tilde{F}_i}$, generalized forces.

5. Evaluate $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \dot{\tilde{p}}_i$ and write down the e.o.m $\boxed{\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}$ for each q_i. $\boxed{\tilde{F}_i = \dot{\tilde{p}}_i}$

6. Identify conserved quantities and solve diff. eqs.

Lagrangian Mechanics cont.

Example: The Point Grey Pendulum



a) Choose generalized coordinates and write down the Lagrangian

b) Write down the equations of motion.

c) Easier or ?