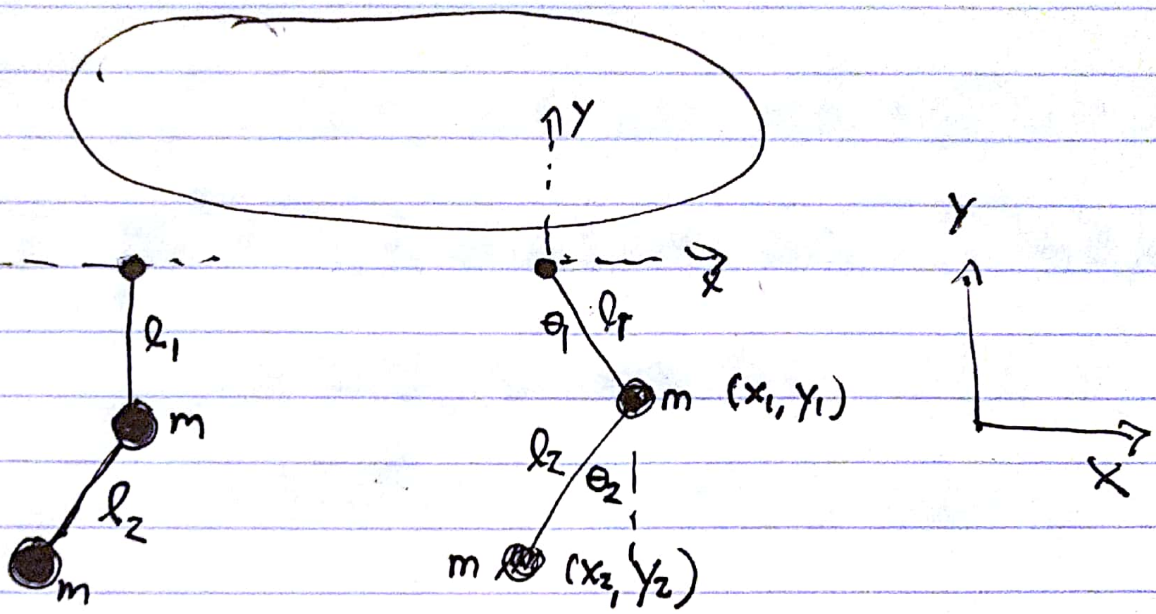
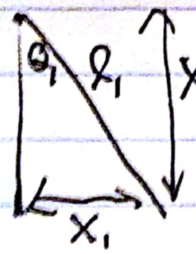
Double Pendulum



KE. $$T = \frac{1}{2} m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2)$$



$$x_1 = l_1 \sin\theta_1 \qquad y_1 = -l_1 \cos\theta_1$$

$$\dot{x}_1 = (l_1 \cos\theta_1)\,\dot{\theta}_1 \qquad \dot{y}_1 = (l_1 \sin\theta_1)\,\dot{\theta}_1$$

$$x_2 = x_1 + l_2 \sin\theta_2 \qquad y_2 = y_1 - l_2 \cos\theta_2$$

$$\dot{x}_2 = \dot{x}_1 + (l_2 \cos\theta_2)\,\dot{\theta}_2 \qquad \dot{y}_2 = \dot{y}_1 + (l_2 \sin\theta_2)\,\dot{\theta}_2$$

$$T = \frac{1}{2} m\Big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_1^2 + 2\dot{x}_1(l_2 \cos\theta_2)\dot{\theta}_2 + (l_2^2 \cos^2\theta_2)\dot{\theta}_2^2 + \dot{y}_1^2 + 2\dot{y}_1(l_2 \sin\theta_2)\dot{\theta}_2 + (l_2^2 \sin^2\theta_2)\dot{\theta}_2^2\Big)$$

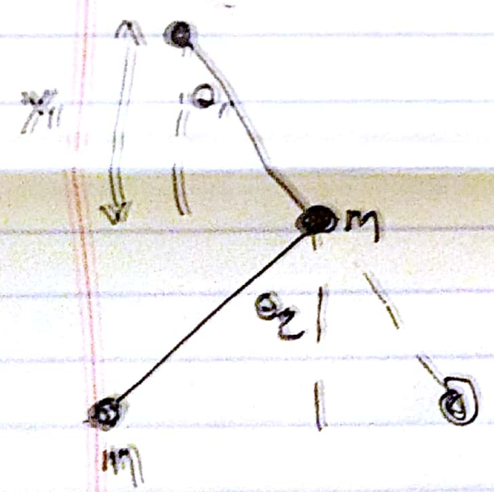
$$T = \frac{1}{2}(2m)(\dot{x}_1^2 + \dot{y}_1^2) + l_2^2 \dot{\theta}_2^2 \left(\frac{1}{2} m\right) + \ldots$$

$$\dot{x}_2^{\,2} = \dot{x}_1^{\,2} + (2 l_1 l_2 \cos\theta_1 \cos\theta_2)\dot\theta_1 \dot\theta_2 + (l_2^2 \cos^2\theta_2)\dot\theta_2^{\,2}$$

$$\dot{y}_2^{\,2} = \cancel{\dot{y}_1}^{\,2} + (2 l_1 l_2 \sin\theta_1 \sin\theta_2)\dot\theta_1 \dot\theta_2 + (l_2^2 \sin^2\theta_2)\dot\theta_2^{\,2}$$

$$\cos A \cos B + \sin A \sin B = \cos(A - B)$$

$$T = m l_1^2 \dot\theta_1^{\,2} + \frac{m}{2} l_2^2 \dot\theta_2^{\,2} + m l_1 l_2 \cos(\theta_1 - \theta_2)\dot\theta_1 \dot\theta_2$$



$$U = mg x_1 + mg y_2$$

$$U = -2mg l_1 \cos\theta_1 - mg l_2 \cos\theta_2$$

$$L = T - U$$

$$\boxed{\begin{aligned} L = {} & m l_1^2 \dot\theta_1^{\,2} + \tfrac{1}{2} m l_2^2 \dot\theta_2^{\,2} + m l_1 l_2 \cos(\theta_1 - \theta_2)\dot\theta_1 \dot\theta_2 \\ & + 2mg l_1 \cos\theta_1 + mg l_2 \cos\theta_2 \end{aligned}}$$

θ_1:

$$\frac{\partial L}{\partial \theta_1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)$$

$$-m l_1 l_2 \sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + (-2mg l_1 \sin\theta_1)$$

$$= \frac{d}{dt}\left(2m l_1^2 \dot{\theta}_1 + m l_1 l_2 \cos(\theta_1-\theta_2)\dot{\theta}_2\right)$$

$$\ddot{\theta}_1 + \frac{l_2}{2l_1}\cos(\theta_1-\theta_2)\ddot{\theta}_2 + \frac{l_2}{2l_1}\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + \frac{g}{l_1}\sin\theta_1 = 0$$

θ_2:

$$\frac{\partial L}{\partial \theta_2} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}\cos(\theta_1-\theta_2)\ddot{\theta}_1 + \frac{l_1}{l_2}\sin(\theta_1-\theta_2)\dot{\theta}_1^2 + \frac{g}{l_2}\sin\theta_2 = 0$$

Near Equilibrium $\theta_1 = \theta_2 = 0$

$$\cos(\theta_1-\theta_2) \simeq 1 + (\sim)$$

$$\sin\theta_1 \simeq \theta_1 + \sim \quad \sin\theta_2 \simeq \theta_2 + \sim$$

$$\sin(\theta_1-\theta_2) \simeq \theta_1-\theta_2 + \cdots$$

Linearized Equations:

$$\ddot{\theta}_1 + \frac{l_2}{2l_1}\cos(\theta_1-\theta_2)\,\ddot{\theta}_2 + \frac{l_2}{2l_1}\sin(\theta_1-\theta_2)\,\dot{\theta}_2^{\,2} + \frac{g}{l_1}\sin\theta_1 = 0$$

$\theta_1, \theta_2 \ll 1$

$$\ddot{\theta}_1 + \frac{l_2}{2l_1}\ddot{\theta}_2 + \frac{g}{l_1}\theta_1 = 0$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}\ddot{\theta}_1 + \frac{g}{l_2}\theta_2 = 0$$

Define. $\gamma = \frac{l_2}{l_1}$ $\qquad \omega_0^2 = \frac{g}{l_1}$

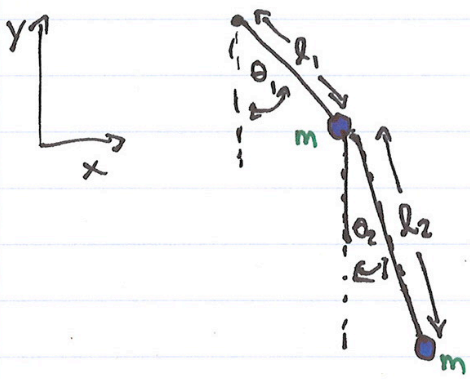
$$\ddot{\theta}_1 + \frac{1}{2}\gamma\ddot{\theta}_2 + \omega_0^2\theta_1 = 0$$

$$\ddot{\theta}_2 + \gamma^{-1}\ddot{\theta}_1 + \gamma^{-1}\omega_0^2\theta_2 = 0$$

$$\ddot{\theta}_1 + \gamma\ddot{\theta}_2 + \omega_0^2\theta_2 = 0$$

Lagrangian Mechanics cont.

Example: Double Pendulum



We can completely characterize this system with the angles θ_1 and θ_2

Kinetic Energy $\quad T = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2)$

$$x_1 = l_1 \sin\theta_1 \qquad \dot{x}_1 = + l_1 \cos\theta_1 \, \dot{\theta}_1$$
$$y_1 = -l_1 \cos\theta_1 \qquad \dot{y}_1 = + l_1 \sin\theta_1 \, \dot{\theta}_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \qquad \dot{x}_2 = \dot{x}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2$$
$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \qquad \dot{y}_2 = \dot{y}_1 + l_2 \sin\theta_2 \, \dot{\theta}_2$$

$$\Rightarrow \dot{x}_2^2 = \dot{x}_1^2 + 2 l_1 l_2 \cos\theta_1 \overset{\times\dot{\theta}_1\dot{\theta}_2}{\cos\theta_2} + l_2^2 \cos^2\theta_2 \, \dot{\theta}_2^2$$
$$\dot{y}_2^2 = \dot{y}_1^2 + 2 l_1 l_2 \sin\theta_1 \underset{\times\dot{\theta}_1\dot{\theta}_2}{\sin\theta_2} + l_2^2 \sin^2\theta_2 \, \dot{\theta}_2^2$$

$$\Rightarrow T = m l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2 + m l_1 l_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2$$

Lagrangian Mechanics cont.

Example: The Double ~~Single~~ Pendulum cont.

Potential Energy $U = mgy_1 + mgy_2$

$$\Rightarrow U = -2mgl_1\cos\theta_1 - mgl_2\cos\theta_2$$

so $$L = ml_1^2\dot{\theta}_1^2 + \frac{1}{2}ml_2^2\dot{\theta}_2^2 + ml_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + 2mgl_1\cos\theta_1 + mgl_2\cos\theta_2$$

Check $l_2 \to 0$ gives simple pendulum of mass $2m$

θ_1: $$\frac{\partial L}{\partial \theta_1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)$$

$$\tilde{F}_1 = \frac{\partial L}{\partial \theta_1} = -ml_1l_2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 - 2mgl\sin\theta_1$$

$$\tilde{P}_1 = \frac{\partial L}{\partial \dot{\theta}_1} = 2ml_1^2\dot{\theta}_1 + ml_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_2$$

$$\Rightarrow \ddot{\theta}_1 + \frac{l_2}{2l_1}\cos(\theta_1-\theta_2)\ddot{\theta}_2 + \frac{l_2}{2l_1}\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + \frac{g}{l_1}\sin\theta_1 = 0$$

Lagrangian Mechanics cont.

Example: The Double Pendulum cont.

θ_2: $$\frac{\partial L}{\partial \theta_2} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)$$

$$\tilde{F}_2 = \frac{\partial L}{\partial \theta_2} = +m l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 - mg l_2 \sin\theta_2$$

$$\tilde{p}_2 = \frac{\partial L}{\partial \dot{\theta}_2} = m l_2^2 \dot{\theta}_2 + m l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1$$

$$\Rightarrow \quad \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + \frac{l_2}{l_1}\ddot{\theta}_2 - \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 + \frac{g}{l_2}\sin\theta_2 = 0$$

Equations of Motion:

$$\ddot{\theta}_1 + \frac{l_2}{2l_1}\cos(\theta_1 - \theta_2)\ddot{\theta}_2 + \frac{l_2}{2l_1}\sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + \frac{g}{l_1}\sin(\theta_1) = 0$$

$$\ddot{\theta}_2 + \frac{l_1}{l_2}\cos(\theta_1 - \theta_2)\ddot{\theta}_1 - \frac{l_1}{l_2}\sin(\theta_1 - \theta_2)\dot{\theta}_1^2 + \frac{g}{l_2}\sin(\theta_2) = 0$$

coupled nonlinear second order ODE's

Lagrangian Mechanics cont.

Example: The Double Pendulum cont.

Check: $\theta_1 = \theta_2 = 0$

$$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 1 \qquad \sin(\theta_1) = \sin(\theta_2) = \sin(\theta_1 - \theta_2) = 0$$

$$\Rightarrow \quad \ddot{\theta}_1 + \frac{l_2}{2l_1}\ddot{\theta}_2 = 0 = \ddot{\theta}_2 + \frac{l_1}{l_2}\ddot{\theta}_1$$

$$\Rightarrow \quad \ddot{\theta}_1 = \ddot{\theta}_2 = 0$$

Equilibrium Point @ $\theta_1 = \theta_2 = 0$

Small Oscillations about Equilibrium

Near Equilibrium $\theta_1 \ll 1 \quad \theta_2 \ll 1$

To linear order

$$\cos(\theta_1 - \theta_2) \simeq 1 + o(\theta_1 - \theta_2)^2$$
$$\sin(\theta_1 - \theta_2) \simeq \theta_1 - \theta_2 + \cdots$$
$$\sin(\theta_1) \simeq \theta_1 + \cdots$$
$$\sin(\theta_2) \simeq \theta_2 + \cdots$$