# Lagrangian Mechanics cont.

## Single Unconstrained Particle

Need 3 $\begin{cases} \text{generalized coordinates} & q_1, q_2, q_3 \\ \text{generalized velocities} & \dot{q}_1, \dot{q}_2, \dot{q}_3 \end{cases}$

to specify the state of the system

From these we can construct the..

Kinetic Energy $\quad T = T(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$

Potential Energy $\quad U = U(q_1, q_2, q_3, t)$

Lagrangian $\quad L = T - U = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)$

Action $\quad S = \int_{t_1}^{t_2} dt\, L$ stationary w.r.t variations of $q_i(t)$

The Path Followed By The System Satisfies

Lagrange's Equations
$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad i=1,2,3$$

for ANY choice of $q_1, q_2, q_3$

# Lagrangian Mechanics cont.

## Single Unconstrained Particle cont.

Any choice of the $q_i$ are equivalent.

In particular, for $(q_1, q_2, q_3) = (x, y, z)$ cartesian coordinates

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$U = U(x, y, z) \quad \text{or} \quad U(x, y, z, t)$$
$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z, t)$$

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p_x$$

So $$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$ Lagrange's Equations

$$\Updownarrow$$

$$F_x = \dot{p}_x$$ Newton's Equations

# Lagrangian Mechanics cont.

## Single Unconstrained Particle cont

More generally we say

$$\tilde{F}_{q_i} = \frac{\partial L}{\partial q_i} \qquad \text{Generalized Force w.r.t. } q_i$$

$$\tilde{p}_{q_i} = \frac{\partial L}{\partial \dot{q}_i} \qquad \text{Generalized Momentum w.r.t } q_i$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad \text{Lagrange's Equations}$$

$$\Updownarrow$$

$$\tilde{F}_{q_i} = \dot{\tilde{p}}_{q_i} \qquad \text{"Generalized" Newton's Equations}$$

Note: If the $q_i$ does not have dimensions of distance then $\tilde{F}_{q_i}$ and $\tilde{p}_{q_i}$ do NOT have dimensions of force or momentum.

e.g. if $q_1 = \theta$ $\quad \tilde{p}_\theta$ is angular momentum
$\tilde{F}_\theta$ is torque

## Lagrangian Mechanics cont.

### Many Unconstrained Particles

Need 3N $\begin{cases} \text{generalized coordinates} & q_1, q_2, \ldots, q_{3N} \\ \text{generalized velocities} & \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{3N} \end{cases}$

to specify the state of the system.

Note: Need not be the "absolute" coordinates of each particle

From these we can construct ...

Kinetic Energy $\quad T = \sum_i T_i = T(q_1, q_2, \ldots, q_{3N}, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{3N})$

Potential Energy $\quad U = U(q_1, q_2, \ldots, q_{3N}, t)$

Lagrangian $\quad L = T - U = L(q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots, q_{3N}, \dot{q}_{3N}, t)$

Action $\quad S = \int_{t_1}^{t_2} dt\, L \quad$ stationary w.r.t variations of the $q_i(t)$

The Path Followed By The System Satisfies

Lagrange's Equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad i = 1, 2, \ldots, 3N$$

for ANY choice of $q_1, q_2, \ldots, q_{3N}$

# Lagrangian Mechanics cont.

## Many Unconstrained Particles cont.

Any choice of the $q_i$ are equivalent.

In particular, for cartesian coordinates

$$\begin{array}{ccccccc} q_1, & q_2, & q_3, & \ldots, & q_{3N-2}, & q_{3N-1}, & q_{3N} \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ x_1, & y_1, & z_1 & \ldots, & x_N, & y_N, & z_N \end{array}$$

$$T = \sum_{i=1}^{N} \frac{1}{2} m \left( \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right)$$

$$U = U(x_1, y_1, z_1, \ldots, x_N, y_N, z_N, t)$$

$$L = \sum_{i=1}^{N} \frac{1}{2} m \left( \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right) - U(x_1, y_1, z_1, \ldots, x_N, y_N, z_N, t)$$

For each coordinate...

$$\frac{\partial L}{\partial x_1} = -\frac{\partial U}{\partial x_1} = F_{x_1}$$

$$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 = p_{x_1}$$

So $$\frac{\partial L}{\partial x_1} = \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{x}_1} \right)$$ Lagrange's Equations

$$\Updownarrow$$

$$F_{x_1} = \dot{p}_{x_1}$$ Newton's Equations

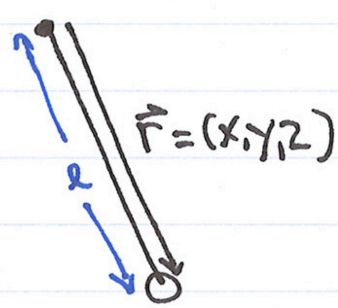
# Lagrangian Mechanics cont.

## Single Particle with Constraints

The primary utility of Lagrangian mechanics is that it can easily handle systems with constraints that might lead to horendous complications in the Newtonian Framework.

While a single particle requires 3 numbers to specify its position it only is necessary to specify $3 - N_c$ degrees of freedom to characterize a system with $N_c$ constraints. We only require $3 - N_c$ generalized coordinates.

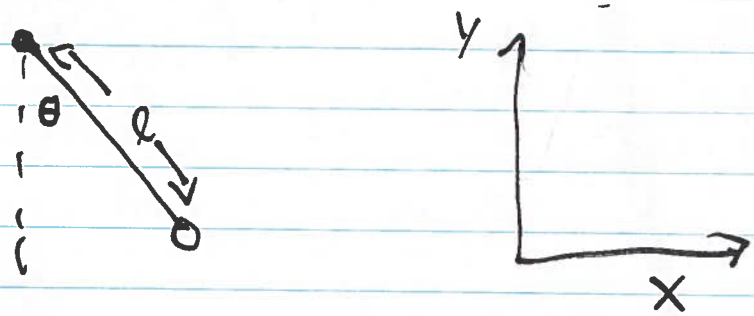
Example: The Simple Pendulum




Position of Bob

$$\vec{r} = (x(t), y(t), z(t))$$

# Example: The Simple Pendulum




$z=0$
1

$$x^2+y^2=\ell^2 \qquad 2$$

$N_c = 2$ constraints

⇒ Only 1 coordinate to describe system. Choose $\theta$ to describe system.

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2$$

$$\vec{r} = (\ell\sin\theta, -\ell\cos\theta, 0) = (x, y, z)$$

$$U = mgy = -mg\ell\cos\theta$$

$$L = T - U = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

$$\frac{\partial L}{\partial\theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) \qquad -mg\ell\sin\theta = \frac{d}{dt}(m\ell^2\dot{\theta})$$

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$T = \frac{1}{2}m(\dot{r}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2)$$

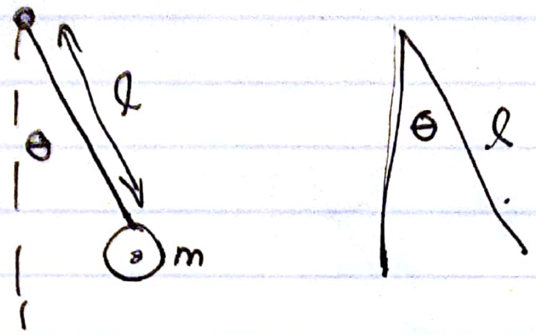
$x = r\cos\theta$ $\quad\quad$ $y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$

$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)$$

Simple




$$T = \frac{1}{2} m \ell^2 \dot{\theta}^2$$

$$U = - mg\ell \cos\theta$$

$$L = T - U = \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell\cos\theta$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

$$-mg\ell\sin\theta = \frac{d}{dt}(m\ell^2\dot{\theta})$$

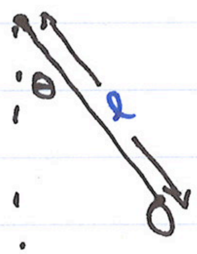
$$m\ell^2\ddot{\theta} = -mg\ell\sin\theta$$

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

# Lagrangian Mechanics cont.

## Single Particle with Constraints cont.

Example: The Simple Pendulum




Constraints: $z=0$ w.l.o.g

$x^2+y^2=\ell^2$

$N_c=2$

$\Rightarrow$ Only require 1 generalized coordinate to describe the system. Can choose this to be $\theta$.

$$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2$$

since $\vec{r}=(x,y,z)=(\ell\sin\theta, -\ell\cos\theta, 0)$

$$U = mgy = -mg\ell\cos\theta$$

so $$L=T-U=\frac{1}{2}m\ell^2\dot{\theta}^2+mg\ell\cos\theta$$

$\Rightarrow$ $$\frac{\partial L}{\partial\theta}=\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)$$

$$-mg\ell\sin\theta=\frac{d}{dt}\left(m\ell^2\dot{\theta}\right)$$

$$\therefore\quad \ddot{\theta}=-\frac{g}{\ell}\sin\theta$$

Example: Double Pendulum