Problem 5.4

(a)

$$1 = \int |\psi_\pm|^2 d^3\mathrm{r}_1 d^3\mathrm{r}_2$$

$$= |A|^2 \int \left[\psi_a(\mathrm{r}_1)\psi_b(\mathrm{r}_2) \pm \psi_b(\mathrm{r}_1)\psi_a(\mathrm{r}_2)\right]^* \left[\psi_a(\mathrm{r}_1)\psi_b(\mathrm{r}_2) \pm \psi_b(\mathrm{r}_1)\psi_a(\mathrm{r}_2)\right] d^3\mathrm{r}_1 d^3\mathrm{r}_2$$

$$= |A|^2 \left[\int |\psi_a(\mathrm{r}_1)|^2 d^3\mathrm{r}_1 \int |\psi_b(\mathrm{r}_2)|^2 d^3\mathrm{r}_2 \pm \int \psi_a(\mathrm{r}_1)^*\psi_b(\mathrm{r}_1) d^3\mathrm{r}_1 \int \psi_b(\mathrm{r}_2)^*\psi_a(\mathrm{r}_2) d^3\mathrm{r}_2\right.$$

$$\left.\pm \int \psi_b(\mathrm{r}_1)^*\psi_a(\mathrm{r}_1) d^3\mathrm{r}_1 \int \psi_a(\mathrm{r}_2)^*\psi_b(\mathrm{r}_2) d^3\mathrm{r}_2 + \int |\psi_b(\mathrm{r}_1)|^2 d^3\mathrm{r}_1 \int |\psi_a(\mathrm{r}_2)|^2 d^3\mathrm{r}_2\right]$$

$$= |A|^2(1\cdot 1 \pm 0\cdot 0 \pm 0 \cdot 0 + 1\cdot 1) = 2|A|^2 \Longrightarrow \boxed{A = 1/\sqrt{2}.}$$

(b)

$$1 = |A|^2 \int \left[2\psi_a(\mathrm{r}_1)\psi_a(\mathrm{r}_2)\right]^* \left[2\psi_a(\mathrm{r}_1)\psi_a(\mathrm{r}_2)\right] d^3\mathrm{r}_1 d^3\mathrm{r}_2$$

$$= 4|A|^2 \int |\psi_a(\mathrm{r}_1)|^2 d^3\mathrm{r}_1 \int |\psi_a(\mathrm{r}_2)|^2 d^3\mathrm{r}_2 = 4|A|^2. \qquad \boxed{A = 1/2.}$$

Problem 5.5

(a)

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_2^2} = E\psi} \quad (\text{for } 0 \le x_1, x_2 \le a, \text{ otherwise } \psi = 0).$$

$$\psi = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$\frac{d^2\psi}{dx_1^2} = \frac{\sqrt{2}}{a}\left[-\left(\frac{\pi}{a}\right)^2\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \left(\frac{2\pi}{a}\right)^2\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$\frac{d^2\psi}{dx_2^2} = \frac{\sqrt{2}}{a}\left[-\left(\frac{2\pi}{a}\right)^2\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \left(\frac{\pi}{a}\right)^2\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$\left(\frac{d^2\psi}{dx_1^2}+\frac{d^2\psi}{dx_2^2}\right)=-\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{a}\right)^2\right]\psi=-5\frac{\pi^2}{a^2}\psi,$$

$$-\frac{\hbar^2}{2m}\left(\frac{d^2\psi}{dx_1^2}+\frac{d^2\psi}{dx_2^2}\right)=\frac{5\pi^2\hbar^2}{2ma^2}\psi=E\psi,\quad \text{with}\quad E=\frac{5\pi^2\hbar^2}{2ma^2}=5K.\quad \checkmark$$

(b) Distinguishable:

$$\boxed{\psi_{22}=(2/a)\sin(2\pi x_1/a)\sin(2\pi x_2/a),\ \text{with } E_{22}=8K}\quad \text{(nondegenerate).}$$

$$\boxed{\left.\begin{aligned}\psi_{13}&=(2/a)\sin(\pi x_1/a)\sin(3\pi x_2/a)\\ \psi_{31}&=(2/a)\sin(3\pi x_1/a)\sin(\pi x_2/a)\end{aligned}\right\},\ \text{with } E_{13}=E_{31}=10K}\quad \text{(doubly degenerate).}$$

Identical Bosons:

$$\boxed{\psi_{22}=(2/a)\sin(2\pi x_1/a)\sin(2\pi x_2/a),\ E_{22}=8K}\quad \text{(nondegenerate).}$$

$$\boxed{\psi_{13}=(\sqrt{2}/a)\left[\sin(\pi x_1/a)\sin(3\pi x_2/a)+\sin(3\pi x_1/a)\sin(\pi x_2/a)\right],\ E_{13}=10K}\quad \text{(nondegenerate).}$$

Identical Fermions:

$$\boxed{\psi_{13}=(\sqrt{2}/a)\left[\sin\left(\tfrac{\pi x_1}{a}\right)\sin\left(\tfrac{3\pi x_2}{a}\right)-\sin\left(\tfrac{3\pi x_1}{a}\right)\sin\left(\tfrac{\pi x_2}{a}\right)\right],\ E_{13}=10K}\quad \text{(nondegenerate).}$$

$$\boxed{\psi_{23}=(\sqrt{2}/a)\left[\sin\left(\tfrac{2\pi x_1}{a}\right)\sin\left(\tfrac{3\pi x_2}{a}\right)-\sin\left(\tfrac{3\pi x_1}{a}\right)\sin\left(\tfrac{2\pi x_2}{a}\right)\right],\ E_{23}=13K}\quad \text{(nondegenerate).}$$

Problem 5.7

(a) $\boxed{\psi(x_1,x_2,x_3)=\psi_a(x_1)\psi_b(x_2)\psi_c(x_3).}$

(b) $$\psi(x_1,x_2,x_3)=\boxed{\begin{aligned}&\tfrac{1}{\sqrt{6}}[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3)+\psi_a(x_1)\psi_c(x_2)\psi_b(x_3)+\psi_b(x_1)\psi_a(x_2)\psi_c(x_3)\\ &+\psi_b(x_1)\psi_c(x_2)\psi_a(x_3)+\psi_c(x_1)\psi_b(x_2)\psi_a(x_3)+\psi_c(x_1)\psi_a(x_2)\psi_b(x_3)]\end{aligned}}.$$

(c) $$\psi(x_1,x_2,x_3)=\boxed{\begin{aligned}&\tfrac{1}{\sqrt{6}}[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3)-\psi_a(x_1)\psi_c(x_2)\psi_b(x_3)-\psi_b(x_1)\psi_a(x_2)\psi_c(x_3)\\ &+\psi_b(x_1)\psi_c(x_2)\psi_a(x_3)-\psi_c(x_1)\psi_b(x_2)\psi_a(x_3)+\psi_c(x_1)\psi_a(x_2)\psi_b(x_3)]\end{aligned}}.$$

Problem 5.16

(a) $E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$. $\rho = \frac{Nq}{V} = \frac{N}{V} = \frac{\text{atoms}}{\text{mole}} \times \frac{\text{moles}}{\text{gm}} \times \frac{\text{gm}}{\text{volume}} = \frac{N_A}{M} \cdot d$, where N_A is Avogadro's number (6.02×10^{23}), M = atomic mass = 63.5 gm/mol, d = density = 8.96 gm/cm^3.

$$\rho = \frac{(6.02 \times 10^{23})(8.96\ \text{gm/cm}^3)}{(63.5\ \text{gm})} = 8.49 \times 10^{22}/\text{cm}^3 = 8.49 \times 10^{28}/\text{m}^3.$$

$$E_F = \frac{(1.055 \times 10^{-34}\text{J} \cdot \text{s})(6.58 \times 10^{-16}\text{eV} \cdot \text{s})}{(2)(9.109 \times 10^{-31}\text{kg})}(3\pi^2\, 8.49 \times 10^{28}/\text{m}^3)^{2/3} = \boxed{7.04\ \text{eV}.}$$

(b)

$$7.04\ \text{eV} = \frac{1}{2}(0.511 \times 10^6 \text{eV}/c^2)v^2 \Rightarrow \frac{v^2}{c^2} = \frac{14.08}{.511 \times 10^6} = 2.76 \times 10^{-5} \Rightarrow \frac{v}{c} = 5.25 \times 10^{-3},$$

so it's $\boxed{\text{nonrelativistic.}}$ $v = (5.25 \times 10^{-3}) \times (3 \times 10^8) = \boxed{1.57 \times 10^6\ \text{m/s.}}$

(c)

$$T = \frac{7.04\,\text{eV}}{8.62 \times 10^{-5}\,\text{eV/K}} = \boxed{8.17 \times 10^4\,\text{K.}}$$

(d)

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} = \frac{(3\pi^2)^{2/3}(1.055 \times 10^{-34})^2}{5(9.109 \times 10^{-31})}(8.49 \times 10^{28})^{5/3}\text{N/m}^2 = \boxed{3.84 \times 10^{10}\ \text{N/m}^2.}$$
