

- We will consider two types of bosonic particles:
 - "real" i.e. number-conserving: He; Rb, Na, K ...
 - 'emergent' phonons, magnons, ...
- Bose-Einstein statistics: based on commutation relation

$$\begin{aligned} [\alpha_k, \alpha_{k'}^+] &= \delta_{kk'} \\ [\alpha_k, \alpha_{k'}^-] &= [\alpha_k^+, \alpha_{k'}^+] = 0 \end{aligned} \tag{1}$$

This gives the Bose-Einstein distribution: $\bar{n}_k = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$

- Consider non-interacting Hamiltonian

$$H = \sum_k (\epsilon_k - \mu) \alpha_k^+ \alpha_k^- \tag{2}$$

Thermal average of an operator \hat{O} is given by

$$\langle \hat{O} \rangle_p = \frac{1}{Z} \sum_j \langle j | \hat{O} | j \rangle e^{-\beta(E_j - \mu N)} \tag{3}$$

$$Z = \sum_j e^{-\beta(E_j - \mu N)} \quad \text{"partition function"}$$

- Consider $\hat{O} = \hat{n}_k = \alpha_k^+ \alpha_k^-$

$$\begin{aligned} \bar{n}_k &= \langle \hat{n}_k \rangle_p = \frac{1}{Z} \sum_j \langle j | \alpha_k^+ \alpha_k^- e^{-\beta(\hat{H} - \mu \hat{N})} | j \rangle \\ &= \frac{1}{Z} \text{Tr} [\alpha_k^+ \alpha_k^- e^{-\beta(\hat{H} - \mu \hat{N})}] \\ &= \frac{1}{Z} \text{Tr} [\alpha_k^- e^{-\beta(\hat{H} - \mu \hat{N})} \alpha_k^+] \\ &= \frac{1}{Z} \text{Tr} [\alpha_k^- \alpha_k^+ e^{-\beta(\hat{H} - \mu \hat{N})} e^{-\beta(\epsilon_k - \mu)}] \end{aligned} \tag{4}$$

$$\begin{aligned}
 \bar{n}_k &= \frac{1}{Z} \text{Tr} \left[(1 + a_k^\dagger a_k) e^{-\beta(\hat{H} - \mu \hat{N})} \right] e^{-\beta(\epsilon_k - \mu)} \\
 &= (1 + \bar{n}_k) e^{-\beta(\epsilon_k - \mu)} \\
 \Rightarrow \bar{n}_k &= \underbrace{\frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}}_T
 \end{aligned} \tag{5}$$

(Similar derivation holds for fermions, giving Fermi-Dirac distribution function.)

Bose-Einstein condensation

- This occurs for "real" or number-conserving bosons, e.g. He⁴ at low temperature T.
- Consider the total number N as a function of T:

$$N = \sum_k \bar{n}_k = \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}, \quad \epsilon_k = \frac{\hbar^2 k^2}{2m} \tag{6}$$

We know that $\mu \leq 0$ (otherwise we'd have some $\bar{n}_k < 0$), therefore it holds $e^{\beta(\epsilon_k - \mu)} \geq e^{\beta \epsilon_k}$

$$\begin{aligned}
 \rightarrow N &\leq \sum_k \frac{1}{e^{\beta \epsilon_k} - 1} \\
 &= N_0 + \underbrace{\frac{V}{(2\pi)^3} \int_{0^+}^{\infty} \frac{4\pi k^2 dk}{\exp(\beta \frac{\hbar^2 k^2}{2m}) - 1}}_{N'(T)}
 \end{aligned} \tag{7}$$

\uparrow
k=0 term
in the sum

V - Volume

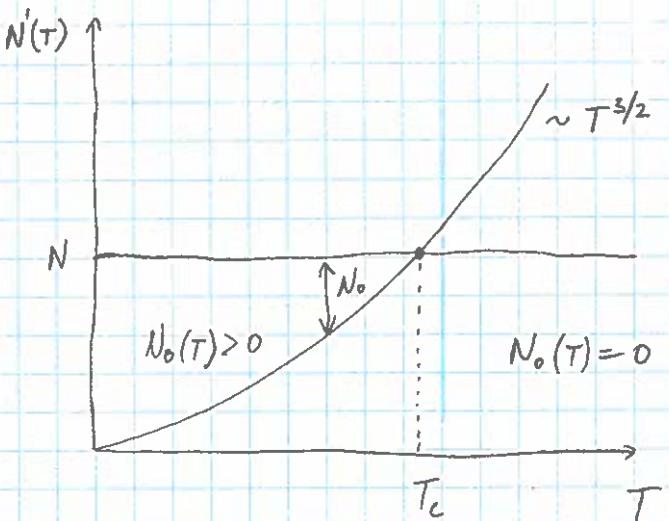
Evaluate $N'(T)$ by substitution $x = \beta \frac{\hbar^2 k^2}{2m}$, $dx = \beta \frac{\hbar^2}{m} k dk$

$$\begin{aligned}
 N'(T) &= \frac{V}{(2\pi)^3} 4\pi \sqrt{2} \left(\frac{m}{\beta \hbar^2} \right)^{3/2} \int_{0^+}^{\infty} \frac{\sqrt{x} dx}{e^x - 1} \\
 &= C V T^{3/2}
 \end{aligned} \tag{8}$$

$$N \leq N_0 + N'(T)$$

\rightarrow For $T < T_c$ an extensive number of bosons must be in the $\vec{k} = 0$ state

BE condensate.



Bogoliubov theory of ${}^4\text{He}$ (1946)

Consider weakly interacting bosons described by

$$\mathcal{H} = \sum_k \varepsilon_k a_k^\dagger a_k + \frac{1}{2} \sum_{k,p,q} V_q a_{k-q}^\dagger a_{p+q}^\dagger a_p a_k \quad (9)$$

Here V_q is a Fourier transform of a short-range interatomic potential, $\varepsilon_k = \hbar^2 k^2 / 2m$.

- Assume V_q is weak and $T \ll T_c$. Then we expect the ground state to be "close to" a perfect BE condensate

$$|\Phi_0^N\rangle = (a_0^\dagger)^N |0\rangle \quad (10)$$

- Following Bogoliubov we approximate

$$\begin{aligned} a_0^\dagger a_0 &\rightarrow \langle a_0^\dagger a_0 \rangle = N_0 \\ a_0^\dagger a_0^\dagger &\rightarrow N_0 \end{aligned} \quad (11)$$

(notice that $a_0^\dagger a_0^\dagger |\Phi_0^N\rangle = \underbrace{\sqrt{(N_0+1)(N_0+2)}}_{\approx N_0 \text{ for large } N_0} |\Phi_0^{N+2}\rangle$)

- In the interaction term we split all sums as

$\sum_k = \sum_{k=0} + \sum'_k$, use Eq. (11) and retain only terms containing at least one power of N_0 :

$$\mathcal{H} \approx \underbrace{\sum_k \epsilon_k a_k^+ a_k}_{k=p=q=0} + \underbrace{\frac{1}{2} N_0^2 V_0}_{p=q=0 \text{ or } k=q=0} + N_0 V_0 \underbrace{\sum'_k a_k^+ a_k}_{p=q=0 \text{ or } k=q=0} + N_0 \underbrace{\sum'_q V_q a_q^+ a_q}_{k-q=p=0} \\ + \underbrace{\frac{1}{2} N_0 \sum'_q V_q (a_q a_{-q} + a_{-q}^+ a_q)}_{k=p=0, k-q=p+q=0} \quad (12)$$

- To simplify define $\epsilon_k \equiv N_0 V_k$, $\hbar \omega_k \equiv \epsilon_k + \zeta_k$

and notice that $N_0 + \underbrace{\sum'_k a_k^+ a_k}_{N'} \approx N_0$, assume $N' \ll N_0$

$$\text{Then } \frac{1}{2} N_0^2 V_0 \approx \frac{1}{2} V_0 [N_0^2 + 2N_0 \sum'_k a_k^+ a_k + \dots] \quad (14)$$

Hence

$$\mathcal{H} = \frac{1}{2} N_0^2 V_0 + \sum_k \left[\hbar \omega_k a_k^+ a_k + \frac{1}{2} \zeta_k (a_k a_{-k} + a_k^+ a_{-k}^+) \right] / \quad (15)$$

↑ "anomalous terms"

- The anomalous terms do not conserve particle number.
This is a consequence of Bogoliubov approximation;
terms $a_k a_{-k}$ represent two bosons with $(\vec{k}, -\vec{k})$ "disappearing
into the condensate."
- Hamiltonian (15) can be diagonalized by means of
the Bogoliubov transformation:

$$\begin{aligned} \alpha_k &= u_k \alpha_k + v_k \alpha_{-k}^+ \\ \alpha_k^+ &= u_k \alpha_k^+ + v_k \alpha_{-k} \end{aligned}$$

$$\begin{aligned} \alpha_k &= u_k \alpha_k - v_k \alpha_{-k}^+ \\ \alpha_k^+ &= u_k \alpha_k^+ - v_k \alpha_{-k} \end{aligned} \quad \text{Inverse} \quad (16)$$

Here α_k are new bosonic "quasiparticle" operators and (u_k, v_k) are real coefficients.

- $[\alpha_k, \alpha_k^+] = 1$ places a constraint on (u_k, v_k) :

$$\begin{aligned} [\alpha_k, \alpha_{k'}^+] &= [u_k \alpha_k - v_k \alpha_{-k}^+, u_{k'} \alpha_{k'}^+ - v_{k'} \alpha_{-k'}] \\ &= u_k u_{k'} [\alpha_k, \alpha_{k'}^+] + v_k v_{k'} [\alpha_{-k}^+, \alpha_{-k'}] \\ &= \delta_{kk'} (u_k^2 - v_k^2) \end{aligned} \quad (17)$$

$$\Rightarrow \underbrace{u_k^2 - v_k^2}_{} = 1 \quad (18)$$

- We want to find (u_k, v_k) that make the resulting Hamiltonian diagonal:

$$H = \sum_k \hbar \omega_k \alpha_k^+ \alpha_k + E_0 \quad \xrightarrow{\text{constant}} \quad (19)$$

$$\begin{aligned} \alpha_k^+ \alpha_k &= (u_k \alpha_k^+ - v_k \alpha_{-k}) (u_k \alpha_k - v_k \alpha_{-k}^+) \\ &= u_k^2 \alpha_k^+ \alpha_k + v_k^2 \alpha_{-k} \alpha_{-k}^+ - u_k v_k (\alpha_k \alpha_{-k} + \alpha_k^+ \alpha_{-k}^+) \end{aligned} \quad (20)$$

- assume $\omega_k = \omega_{-k}$ and $v_k^2 = v_{-k}^2$ (check later), then

$$\begin{aligned} \sum_k \hbar \omega_k \alpha_k^+ \alpha_k &= \sum_k \hbar \omega_k (u_k^2 + v_k^2) \alpha_k^+ \alpha_k + \sum_k \hbar \omega_k v_k^2 \\ &\quad - \sum_k \hbar \omega_k u_k v_k (\alpha_k \alpha_{-k} + \alpha_k^+ \alpha_{-k}^+) \end{aligned} \quad (21)$$

Comparison to Eq. (15) implies:

$$\left. \begin{aligned} \hbar\omega_k (\mu_k^2 + \nu_k^2) &= \hbar\omega_k \\ 2\hbar\omega_k \mu_k \nu_k &= \epsilon_k \end{aligned} \right\} \quad (22)$$

To solve for ω_k we square both equations and subtract:

$$(\hbar\omega_k)^2 \left[\underbrace{(\mu_k^2 + \nu_k^2) - 4\mu_k^2\nu_k^2}_{(\mu_k^2 - \nu_k^2)^2 = 1} \right] = (\hbar\omega_k)^2 - \epsilon_k^2$$

$$\left. \Rightarrow \hbar\omega_k = \sqrt{\hbar^2\omega_k^2 - \epsilon_k^2} \right\} = \sqrt{\epsilon_k(\epsilon_k + 2NV_k)}$$

"Spectrum of
quasiparticle
excitations"

(23)

① Non-interacting bosons ($V_k = 0$)

$$\hbar\omega_k = \sqrt{\epsilon_k^2} = \frac{\hbar k}{2m} \checkmark$$

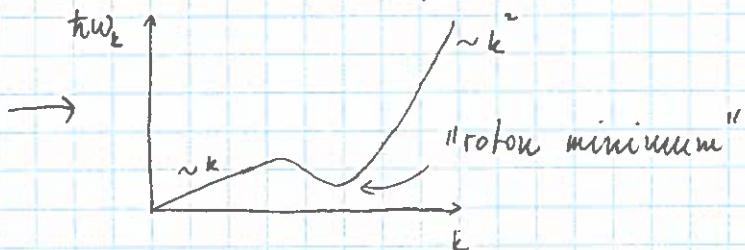
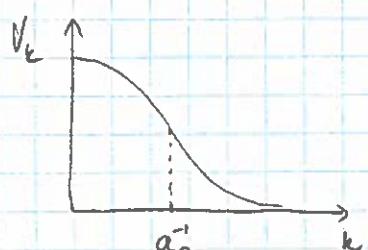
$$H_0 = \sum_k \frac{\hbar^2 k^2}{2m} a_k^+ a_k$$

② Contact repulsion case $V(r) = U\delta(r)$

$$\rightarrow V_k = \frac{U}{V_k} \text{ for all } \vec{k}$$

$$\hbar\omega_k = \sqrt{\epsilon_k \left(\epsilon_k + \frac{2(NU/V_k)}{E_0} \right)} \approx \begin{cases} \sqrt{\epsilon_k E_0} \sim |\vec{k}|, & \text{for } \epsilon_k \ll E_0 \\ (\text{sound-like}) \\ \text{dispersion} \\ |\epsilon_k| \sim k^2, & \text{for } \epsilon_k \gg E_0 \\ (\text{particle-like}) \end{cases} \quad (24)$$

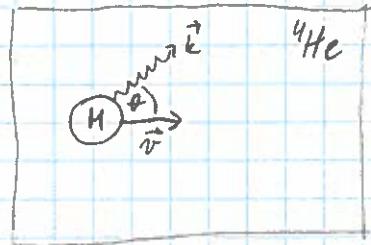
③ Typical "He interaction has finite range a_0



Landau argument for superfluidity in ^4He

- Consider an object with large mass M moving with velocity \vec{v} in liquid ^4He .

Q: Can it relax its energy & momentum by creating quasiparticle excitations?



$$\frac{1}{2}Mv^2 = \frac{1}{2}Mv'^2 + \hbar\omega_k \quad (\text{energy conservation})$$

$$M\vec{v} = M\vec{v}' + \hbar\vec{k} \quad (\text{momentum}) \quad (25)$$

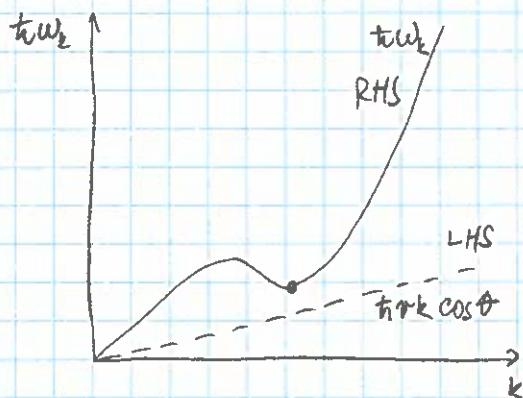
$$M^2v'^2 = M^2v^2 + \hbar^2k^2 - 2M\hbar\vec{k}\cdot\vec{v}$$

Combine two equations:

$$\hbar\vec{v}\cdot\vec{k} = \frac{\hbar^2k^2}{2M} + \hbar\omega_k \quad (26)$$

Consider case when $M \gg m$: we can neglect $\frac{\hbar^2k^2}{2M} = \frac{\hbar^2k^2}{2m} \left(\frac{M}{m}\right)$

- The graph shows that slow-moving object cannot dissipate ENERGY & MOMENTUM
- dissipationless motion
"superflow"



- One can solve for critical velocity:

$$v_c = \min_k \left(\frac{\hbar k}{2M} + \frac{\omega_k}{k} \right) \approx 1 \text{ cm/s in } ^4\text{He} \quad (27)$$