

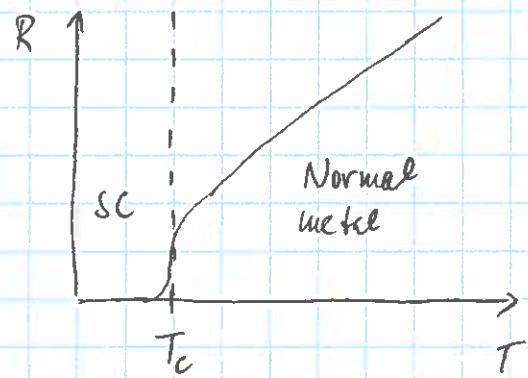
SUPERCONDUCTIVITYBasic phenomenology:

i) Many metals exhibit superconductivity

- a state with ZERO DC resistivity

observed below critical temperature

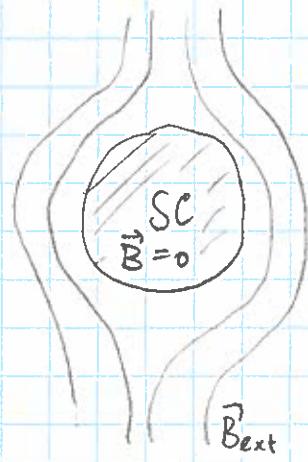
T_c (of order of few degrees Kelvin)



ii) Superconductors behave as perfect diamagnets - a sample in thermal

equilibrium EXPELS applied mag. field from its interior

(Meissner effect)



iii) Electrons in most (but not all)

superconductors have a gap 2Δ

centered around the Fermi energy ϵ_F .

(Read A&H ch 84 for more general info, including persistent currents, critical fields, flux quantization, Josephson effect.)

- The following notes are based on a book "Introduction to Superconductivity" by M. Tinkham, Chapter 2.

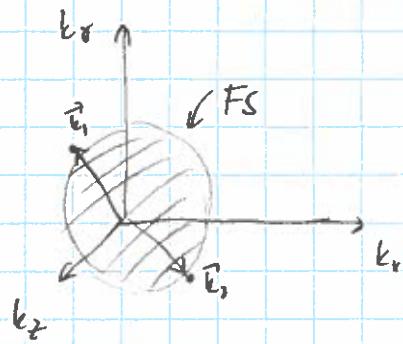
Cooper instability (1956, L.N. Cooper)

- For arbitrarily weak ATTRACTIVE INTERACTION electrons in the Fermi sea are unstable towards formation of PAIRS (e-e bound states, a.k.a. Cooper pairs).
- Consider the "Cooper problem", two electrons subject to attractive interaction in the presence of Fermi sea

$$H = \frac{\vec{p}_1^2}{2m} + \frac{\vec{p}_2^2}{2m} + V(\vec{r}_1 - \vec{r}_2) \quad (1)$$

Assume:

- electrons have total momentum $\vec{k}_1 + \vec{k}_2 = 0$
- they are in spin-singlet state



$$\psi_0(\vec{r}_1, \vec{r}_2) = \sum_{|\vec{k}| > k_F} g_k e^{i\vec{k} \cdot \vec{r}_1} e^{-i\vec{k} \cdot \vec{r}_2} (\uparrow\downarrow - \downarrow\uparrow) / \sqrt{2} \quad (2)$$

↑ unknown function, to be determined
 $g_k = -g_{-k}$ due to Fermi statistics

$$\psi_0(\vec{r}_1 - \vec{r}_2) = \sum_{|\vec{k}| > k_F} g_k \cos \vec{k} \cdot (\vec{r}_1 - \vec{r}_2) (\uparrow\downarrow - \downarrow\uparrow) / \sqrt{2} \quad (3)$$

- Substitute ψ_0 into Schrödinger eq. $H\psi_0 = E\psi_0$ with Hamiltonian (1). We obtain an equation for g_k :

$$(E - 2\epsilon_k) g_k = \sum_{|\vec{k}'| > k_F} V_{kk'} g_{k'} \quad (4)$$

where

$$\epsilon_k = \frac{\hbar^2 k^2}{2m} \quad \text{and} \quad V_{kk'} = \frac{1}{V} \int d^3 r' V(\vec{r}) e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}}$$

↑ volume

We now make the "Cooper ansatz" and take

$$V_{kk'} = \begin{cases} -V & \text{when } |\epsilon_k - \epsilon_F|, |\epsilon_{k'} - \epsilon_F| < t\omega_C \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Eq. (4) then becomes

$$(E - 2\epsilon_k) g_k = -V \sum_{k'}^{\textcircled{1}} g_{k'} \quad \begin{matrix} \text{restriction to } |k'| > k_F \text{ &} \\ |\epsilon_{k'} - \epsilon_F| < t\omega_C \end{matrix} \quad (6a)$$

$$\rightarrow \boxed{g_k = V \frac{\sum_{k'}^{\textcircled{1}} g_{k'}}{2\epsilon_k - E}} \quad (6b)$$

Now sum both sides $\sum_k^{\textcircled{1}}$ to obtain

$$\left(\frac{1}{V} = \sum_k^{\textcircled{1}} \frac{1}{2\epsilon_k - E} \right) \quad (7)$$

Convert into an integral and solve for E

$$\begin{aligned} \frac{1}{V} &= \int_{E_F}^{E_F + t\omega_C} d\epsilon N(\epsilon) \frac{1}{2\epsilon - E} \approx N(\epsilon_F) \int_{E_F}^{E_F + t\omega_C} \frac{d\epsilon}{2\epsilon - E} \\ &= \underbrace{\frac{1}{2} N(\epsilon_F) \ln \left[1 + \frac{2t\omega_C}{2\epsilon_F - E} \right]}_{(8)} \end{aligned}$$

$$\rightarrow \boxed{\frac{1}{1 + \frac{2t\omega_C}{2\epsilon_F - E}} = e^{2/VN(\epsilon_F)}} \quad (9)$$

Make a weak coupling assumption: $VN(\epsilon_F) \ll 1$

$$\boxed{E \approx 2\epsilon_F - 2t\omega_C e^{-2/VN(\epsilon_F)}} \quad (10)$$

- the two-electron bound state has energy lower than $2\epsilon_F$
- it is a stable state
- result non-analytic in $V \Rightarrow$ Cooper pairing is a non-perturbative phenomenon!

The BCS ground state

(Bardeen, Cooper, Schrieffer, 1957)

- What happens for many electrons in the presence of attractive interaction?

$$\left[\begin{array}{l} |\Psi_c\rangle = \prod_k (\mu_e + v_e c_{k\uparrow}^+ c_{-k\downarrow}^+) |0\rangle \\ |\mu_e|^2 + |v_e|^2 = 1 \end{array} \right] \xrightarrow{\text{BCS wavefunction}} \quad (1)$$

- $|v_e|^2$ - probability that a pair state $(k\uparrow, -k\downarrow)$ is occupied
 - Note that $|\Psi_c\rangle$ is a superposition of states with different electron numbers!
-

To calculate (μ_e, v_e) we consider the reduced BCS (pairing) Hamiltonian:

$$H - \mu N = \sum_{k\uparrow\downarrow} (\epsilon_k - \mu) c_{k\uparrow}^+ c_{k\downarrow} + \sum_{k,e} V_{ke} c_{e\uparrow}^+ c_{-k\downarrow}^+ c_{-e\downarrow} c_{e\uparrow} \quad (12)$$

Employ variational principle, minimize the g.s. energy

$$E_s = \langle \Psi_c | H - \mu N | \Psi_c \rangle \quad (13)$$

with respect to μ_e, v_e . Evaluate E_s :

$$E_s = 2 \sum_e \int_k |v_e|^2 + \sum_{k,e} V_{ke} \mu_e v_e^* \mu_e^* v_e \quad (14)$$

where $\int_k = \epsilon_k - \mu$. Assume $\mu_e, v_e \in \mathbb{R}$ and write

$$\mu_e = \sin \theta_e, \quad v_e = \cos \theta_e \quad (\mu_e^2 + v_e^2 = 1) \quad (15)$$

$$\rightarrow E_s = \sum_e \int_k (1 + \cos 2\theta_e) + \frac{1}{4} \sum_{k,e} V_{ke} \sin 2\theta_e \sin 2\theta_e \quad (16)$$

Minimize with respect to θ_e :

$$0 = \frac{\partial E_k}{\partial \theta_k} = -2\sum_e V_{ke} \sin 2\theta_e + \cos 2\theta_k \sum_e \sin 2\theta_e$$

$$\rightarrow \tan 2\theta_k = \frac{\sum_e V_{ke} \sin 2\theta_e}{2\sum_e \cos 2\theta_e} = -\frac{\Delta_k}{\xi_k} \quad (17)$$

We define:

$$\Delta_k = -\sum_e V_{ke} \mu_e v_e = -\frac{1}{2} \sum_e V_{ke} \sin 2\theta_e \quad (18)$$

$$E_k = \sqrt{\Delta_k^2 + \xi_k^2}$$

Solve using trig. identities:

$$2\mu_e v_e = \sin 2\theta_e = (1 + \tan^2 2\theta_e)^{-1/2} = \frac{\Delta_k}{E_k} \quad (19)$$

$$v_e^2 - \mu_e^2 = \cos 2\theta_e = -(1 + \tan^2 2\theta_e)^{-1/2} = -\frac{\xi_k}{E_k}$$

$$\Rightarrow \left. \begin{array}{l} \mu_e^2 = \frac{1}{2} \left(1 + \frac{\xi_k}{E_k} \right) \\ v_e^2 = \frac{1}{2} \left(1 - \frac{\xi_k}{E_k} \right) \end{array} \right| \quad (20)$$

The sign is chosen so that $v_e^2 \rightarrow 0$, $\mu_e^2 \rightarrow 1$ as $\xi_k \rightarrow +\infty$.

In the above solution Δ_k is still undetermined. To solve for Δ_k substitute (19) into (18):

$$\left[\Delta_k = -\frac{1}{2} \sum_e V_{ke} \frac{\Delta_k}{E_k} = -\frac{1}{2} \sum_e V_{ke} \frac{\Delta_k}{\sqrt{\Delta_k^2 + \xi_k^2}} \right] \quad (21)$$

This is often called "BCS gap equation" - we shall see that Δ_k has interpretation as excitation gap.

- to solve we use the Cooper ansatz Eq. (5) for V_{ke} and obtain:

$$\Delta_\epsilon = \begin{cases} \Delta & \text{for } |\xi_\epsilon| < \omega_c \\ 0 & \text{otherwise} \end{cases} \quad (22)$$

where

$$\Delta = \frac{1}{2} V \sum_{\epsilon} \frac{\Delta}{\sqrt{\xi_\epsilon^2 + \Delta^2}} \quad |\xi_\epsilon| < \omega_c \quad (23)$$

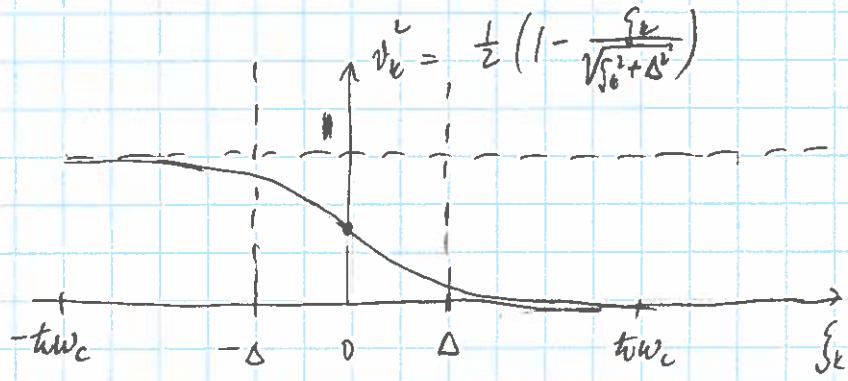
When $\Delta \neq 0$ we have

$$\frac{2}{V} = \sum_{\epsilon} \frac{1}{\sqrt{\xi_\epsilon^2 + \Delta^2}} = \int_{-\omega_c}^{\omega_c} d\xi \frac{N(\xi)}{\sqrt{\xi^2 + \Delta^2}} \quad (24)$$

$$\approx N(0) \int_{-\omega_c}^{\omega_c} d\xi \frac{1}{\sqrt{\xi^2 + \Delta^2}} = 2N(0) \operatorname{Ar sinh} \left(\frac{\omega_c}{\Delta} \right)$$

$$\rightarrow \left[\Delta = \frac{\omega_c}{\sinh [1/VN(0)]} \xrightarrow{VN(0) \ll 1} \frac{2\omega_c e^{-1/N(0)V}}{V} \right] \quad (25)$$

- Once again non-analytic dependence on V signals the non-perturbative nature of the result.



when $\Delta \neq 0$ we have
a "paired state" which
replaces the usual
Fermi sphere as the
ground state of the
system.

Condensation energy

- defined as difference between BCS ground state energy E_g and the energy E_N of the "normal state" (the Fermi sphere)

$$E_s = \langle \Psi_0 | \hat{H} - \mu \hat{N} | \Psi_0 \rangle = \sum_k \left(\xi_k - \frac{\xi_k^2}{E_k^2} \right) - \frac{\Delta^2}{V} \quad (26a)$$

$$E_N = \langle \Psi_0 | \hat{H} - \mu \hat{N} | \Psi_0 \rangle_{\Delta=0} = 2 \sum_{k < k_F} \xi_k \quad (26b)$$

The difference, when simplified, becomes

$$\boxed{E_s - E_N = - \frac{1}{2} N(0) \Delta^2} \quad (27)$$

↗ BCS condensation energy

→ The condensation energy is NEGATIVE which shows that $|\Psi_0\rangle$ is a stable ground state when $V \neq 0$.