

PHONON-MEDIATED ATTRACTION
BETWEEN ELECTRONS

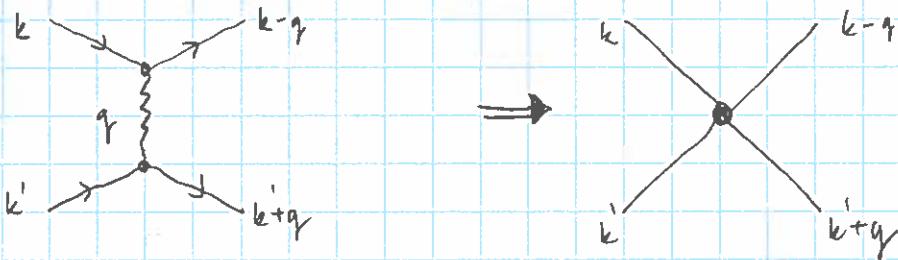
- Surprisingly, phonons can mediate ATTRACTIVE interaction between electrons in a metal. This has important consequences
 \rightarrow the attraction causes SUPERCONDUCTIVITY.

- We sketch a derivation based on Frölich Hamiltonian:

$$\left[\mathcal{H} = \mathcal{H}_0 + \mathcal{H}' = \sum_q w_q a_q^\dagger a_q + \sum_k \varepsilon_k c_k^\dagger c_k \right] \quad (1)$$

$$+ H \sum_{kq} c_{k+q}^\dagger c_k (a_q^\dagger + a_q)$$

- we want to derive an "effective Hamiltonian" for electrons where phonons have been "integrated out." Schematically:



- This can be achieved using the technique of "canonical transformation." Consider Hamiltonian $H = H_0 + \lambda H'$, transform

$$H \rightarrow \tilde{H} = e^{-S} H e^S = H + [H, S] + \frac{1}{2} [[H, S], S] + \dots \quad (2)$$

and find operator S such that \tilde{H} is indep. of λ to linear order.

$$\tilde{H} = H_0 + \underbrace{(\lambda H' + [H_0, S])}_{\text{assume } S \sim \lambda} + \lambda [H', S] + \dots \quad (3)$$

assume $S \sim \lambda$ and demand that this term vanishes

So, we need to find S such that

$$[H_0, S] = -\lambda H' \quad (4)$$

Solve by assuming a basis: $H_0 |\phi_m\rangle = \epsilon_m |\phi_m\rangle$

$$\langle \phi_n | \tilde{H}_0 S - S \tilde{H}_0 | \phi_m \rangle = -\lambda \langle \phi_n | H' | \phi_m \rangle$$

$$(\epsilon_n - \epsilon_m) \langle \phi_n | S | \phi_m \rangle = -\lambda \langle \phi_n | H' | \phi_m \rangle$$

$$\rightarrow \langle \phi_n | S | \phi_m \rangle = \lambda \frac{\langle \phi_n | H' | \phi_m \rangle}{\epsilon_m - \epsilon_n} \quad (5)$$

We can therefore write

$$\tilde{H} = H_0 + \underbrace{\lambda [H', S]}_{O(\lambda^2)} + \frac{1}{2} \underbrace{[(H_0)S], S}_{O(\lambda^3)} + \underbrace{\frac{1}{2} [H', S], S}_{O(\lambda^3)}$$

$$\boxed{\tilde{H} = H_0 + \frac{1}{2} \lambda [H', S] + O(\lambda^3)} \quad (6)$$

• Apply this to the e-ph interaction

- Consider $T=0$, i.e. no thermal phonons

- As basis states we take eigenstates of $H_0^{ph} = \sum_q w_q a_q^\dagger a_q$

$|0\rangle$ (no phonons)

$|\vec{q}\rangle$ (one phonon with momentum \vec{q})

(states with more phonons are unimportant when working to order λ^2).

• Calculate matrix elements of S according to Eq. (5)

$$\langle \vec{q} | S | 0 \rangle = M \sum_{k,p} \frac{\langle \vec{q} | c_{k+p}^\dagger c_k (a_{-q}^\dagger + a_q) | 0 \rangle}{\epsilon_k - \epsilon_{k+p} - w_q} \quad (7)$$

$$= H \sum_k \frac{c_{k-q}^+ c_k}{\epsilon_k - \epsilon_{k-q} - \omega_q}$$

$$\langle 0 | s | \vec{q} \rangle = N^* \sum_k \frac{c_{k+q}^+ c_k}{\epsilon_k - \epsilon_{k+q} + \omega_q} \quad (8)$$

- We now can find the leading $\sim \lambda^2$ correction in \tilde{H} . Eq. (6) in the ground state with zero phonons:

$$\begin{aligned} \langle 0 | [\chi] s | 0 \rangle &= \langle 0 | \chi' s | 0 \rangle - \langle 0 | s \chi' | 0 \rangle \\ &= \sum_q (\langle 0 | \chi' | q \rangle \langle q | s | 0 \rangle - \langle 0 | s | q \rangle \langle q | \chi' | 0 \rangle) \\ &= |H|^2 \sum_q \sum_{k, k'} c_{k+q}^+ c_{k'}^+ c_{k-q}^+ c_k \left(\frac{1}{\epsilon_k - \epsilon_{k-q} - \omega_q} - \frac{1}{\epsilon_{k'} - \epsilon_{k'+q} + \omega_q} \right) \end{aligned} \quad (9)$$

- In the last term reverse $\vec{k} \leftrightarrow \vec{k}'$ and reverse $\vec{q} \rightarrow -\vec{q}$:

$$\boxed{\tilde{\chi} = \chi_0 + |H|^2 \sum_{\substack{k, k' \\ \vec{q}}} \frac{\omega_q}{(\epsilon_k - \epsilon_{k-q})^2 - \omega_q^2} c_{k+q}^+ c_{k'}^+ c_{k-q}^+ c_k} \quad (10)$$

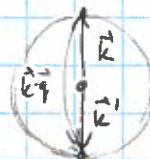
This is the effective electron Hamiltonian.

- The phonon-mediated interaction between electrons is ATTRACTIVE

when

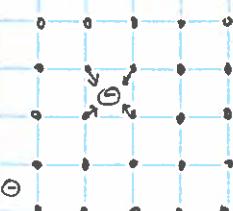
$$|\epsilon_k - \epsilon_{k-q}| < \omega_q \quad (11)$$

$$\rightarrow \frac{\hbar^2}{2m} |2\vec{k} \cdot \vec{q} - \vec{q}^2| < c_s(\vec{q})$$



satisfied whenever $|\vec{k}| \approx k_F$ and $\vec{q} \approx 2\vec{k}$

- Physical picture:



- ions are attracted to the instantaneous electron position creating excess positive charge
- second electron is then attracted to this distortion in the lattice
→ retarded attraction!