

- The transformation, however, is not useful as written due to the $\sqrt{\quad}$ factors in (3).
- For low-energy excited states near the ordered ground state, either FM or AF, we will assume that only a small number of spins are "reversed", that is $S - \langle S_j^z \rangle = \langle a_j^+ a_j \rangle$ is small,

or: $\frac{\langle n_j \rangle}{S} \ll 1$ ← "valid for large S" (6)

Hence we expand the square root factors as follows

$$S^+ = \sqrt{2S} \left[a_j - \frac{n_j}{4S} a_j + \dots \right] \quad (7)$$

$$S^- = \sqrt{2S} \left[a_j^+ - a_j^+ \frac{n_j}{4S} + \dots \right]$$

and retain only the leading term in \hbar . Hence

$$\vec{S}_i \cdot \vec{S}_j = \frac{1}{2} (S_i^+ S_j^- + S_i^- S_j^+) + S_i^z S_j^z \quad (8)$$

$$\approx S (a_i^+ a_j + a_i a_j^+ + S - n_i - n_j) + \dots$$

(I) FERROMAGNETIC CASE ($J > 0$)

$$H = -JS \sum_{\langle i,j \rangle} (a_i^+ a_j + a_i a_j^+ + S - n_i - n_j) - 2B \sum_j (S - n_j) \quad (9)$$

i, j are "nearest neighbor" sites on some d -dimensional lattice

- we solve this by Fourier transforming:

$$b_k = \frac{1}{\sqrt{N}} \sum_j e^{i\vec{k} \cdot \vec{r}_j} a_j \quad (10)$$

$$a_j = \frac{1}{\sqrt{N}} \sum_k e^{-i\vec{k} \cdot \vec{r}_j} b_k$$

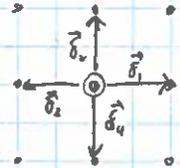
- In momentum space H assumes the form

$$H = -JNzS^2 - 2BNS + \mathcal{H}_0 + \mathcal{H}_1 \quad (11)$$

where z denotes the "coordination number" (the # of nearest neighbors of site j) and

$$\mathcal{H}_0 = \sum_{\mathbf{k}} \left[2JSzS(1-\gamma_{\mathbf{k}}) + 2B \right] b_{\mathbf{k}}^+ b_{\mathbf{k}} \quad (12)$$

with $\gamma_{\mathbf{k}} = \frac{1}{z} \sum_{\vec{\delta}} e^{i\vec{k} \cdot \vec{\delta}}$



(The form of \mathcal{H}_0 is valid for lattices

with an inversion center which implies $\gamma_{\mathbf{k}} = \gamma_{-\mathbf{k}}$)

- \mathcal{H}_1 contains higher-order terms in $b_{\mathbf{k}}, b_{\mathbf{k}}^+$ and represents magnon interactions.

- We may write

$$\mathcal{H}_0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} n_{\mathbf{k}} \quad \text{diagonal Hamiltonian} \quad (13)$$

with $\omega_{\mathbf{k}} = 2JSz(1-\gamma_{\mathbf{k}}) + 2B$
 the "magnon spectrum"

Example: cubic lattice in 3D $\vec{\delta} = \pm a\hat{x}, \pm a\hat{y}, \pm a\hat{z}$ (14)

$$z(1-\gamma_{\mathbf{k}}) = 6 - \sum_{\vec{\delta}} e^{i\vec{k} \cdot \vec{\delta}} = 2(3 - \cos k_x a - \cos k_y a - \cos k_z a)$$

At long wavelengths ($ka \ll 1$) we expand $\cos ka \approx 1 - \frac{1}{2}(ka)^2$ and find

$$\omega_{\mathbf{k}} = 2B + 2JS(ka)^2 \quad (15)$$

- this result also holds for FCC and BCC lattices.

→ in zero magnetic field FM magnons exhibit particle-like dispersion

$$\omega_k = \frac{k^2}{2m^*}, \quad m^* = \frac{1}{4\gamma S a^2} \quad (15)$$

↑
"effective mass"

- for conventional ferromagnets it is found that $m^* \approx 10m_e$.

• Magnon heat capacity

Take $\omega_k = Dk^2$ (with $D = 2\gamma S a^2$) and calculate the internal energy

$$\begin{aligned} U &= \sum_k \omega_k \langle n_k \rangle_T = \sum_k \frac{\omega_k}{e^{\beta\omega_k} - 1} \\ &= \frac{1}{(2\pi)^3} \int_{|\mathbf{k}| < k_{\max}} d^3k Dk^2 \frac{1}{e^{Dk^2\beta} - 1} \\ &= \frac{(k_B T)^{5/2}}{4T^2 D^{3/2}} \int_0^{x_m} dx \frac{x^{3/2}}{e^x - 1} \\ &\approx \frac{0.45}{\pi^2} \frac{(k_B T)^{5/2}}{D^{3/2}} \quad (\text{taking } x_m \rightarrow \infty) \end{aligned} \quad (16)$$

$x = Dk^2\beta$
 $x_m = Dk_{\max}^2\beta$

↑ valid for low T.

Heat capacity: $C_V = \frac{dU}{dT} \approx 0.113 k_B (k_B T / D)^{3/2}$ (17)

• To observe this for an insulating ferromagnet we include phonon contribution $\sim bT^3$ and write

$$\begin{aligned} C_V^{\text{tot}} &\approx c T^{3/2} + b T^3 \quad / T^{3/2} \\ \frac{C_V^{\text{tot}}}{T^{3/2}} &\approx c + b T^{3/2} \end{aligned} \quad (18)$$

Then plot measured $C_V^{\text{tot}} / T^{3/2}$ vs $T^{3/2}$; should see a straight line. This is indeed observed for many ferromagnets. (See Fig. 2 in Kittel handout.)

• Magnetization reversal

- we expect the magnetization $M_s = 2\mu_0 \sum_i \langle S_i^z \rangle$ to DECREASE as T is raised and more magnons are thermally excited:

$$M_s(T) = 2\mu_0 (NS - \sum_k \langle b_k^+ b_k \rangle) \quad (19)$$

We calculate $\Delta H(T) = M_s(0) - M_s(T)$

$$\rightarrow \Delta H(T) = 2\mu_0 \sum_k \langle n_k \rangle = \frac{2\mu_0}{(2\pi)^d} \int d^d k \frac{1}{e^{\beta D k^2} - 1} \quad (20)$$

$$= S_d \int_0^{k_{max}} dk k^{d-1} \frac{1}{e^{\beta D k^2} - 1}$$

$$= S_d \int_0^{x_m} \frac{dx}{\beta D} \left(\frac{x}{\beta D} \right)^{\frac{d-2}{2}} \frac{1}{e^x - 1}$$

$$= S_d \left(\frac{k_B T}{D} \right)^{\frac{d}{2}} \int_0^{x_m} dx \frac{x^{(d-2)/2}}{e^x - 1}$$

$$x = \beta D k^2 \\ dx = 2\beta k dk$$

S_d - surface of a unit sphere in d dimensions

$$S_3 = 4\pi, S_2 = 2\pi, S_1 = 1$$

- Note that because $e^x - 1 \sim x$ for small x the integral DIVERGES at the lower bound when $d \leq 2$. This implies that thermal fluctuations tend to destabilize FM order in low dimensions ($d \leq 2$).
- In $d=3$ we find $\Delta H(T) \sim T^{3/2}$ which is experimentally observed.

READING ASSIGNMENT: "AF Magnons"
p. 58-62 from Kittel handout