P502	Problem set # 6	No due date

1. (10 points) *"Size of the Cooper pair"–* Calculate the mean square radius ρ of the Cooper pair,

$$\rho^2 = \frac{\int d^3r r^2|\Psi(\mathbf{r})|^2}{\int d^3r|\Psi(\mathbf{r})|^2},$$

where $\Psi(\mathbf{r})$ is the pair wavefunction discussed in class in connection with the Cooper problem. Express the answer in terms of v_F and Δ (plus any universal constants you need). Estimate the value of ρ for a typical superconductor with $T_c \simeq 10$K.

Hint: Transform the integrals into k-space sums and then into integrals over energy. Also note the indentity $\nabla_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} = i\mathbf{r}e^{i\mathbf{k}\cdot\mathbf{r}}$.

2. (20 points) *"d-wave superconductivity in high-T_c cuprates"–* Among the early triumphs of the high-T_c field (to which UBC group of Bonn and Hardy made significant contributions) was establishing the unconventional d-wave symmetry of the order parameter Δ. This means that when rotated by 90 degrees the pair wavefunction maps to the negative of itself. This has profound consequences for the electronic properties of cuprates, some of which we now explore in a simple model.

Superconductivity in cuprates is believed to originate in two-dimensional copper-oxygen planes. We consider a simple model of tight binding electrons hopping on a square lattice of copper ions with attraction between electrons on nearest neighbor sites responsible for d-wave superconductivity. The mean-field Hamiltonian describing this situation may be written as

$$H = -t\sum_{\langle ij\rangle\sigma}(c^\dagger_{i\sigma}c_{j\sigma} + \text{h.c.}) - \mu\sum_{i\sigma}c^\dagger_{i\sigma}c_{i\sigma} + \frac{1}{2}\sum_{\langle ij\rangle}[\Delta^*_{ij}(c_{i\downarrow}c_{j\uparrow} - c_{i\uparrow}c_{j\downarrow}) + \text{h.c.}]$$

where Δ_{ij} is an order parameter associated with the nearest neighbor bond. The uniform d-wave state is characterized by $\Delta_{ij} = \pm\Delta_0$ when i, j are nearest neighbors (and zero otherwise). The plus sign occurs for horizontal bonds, minus for vertical bonds; Δ_0 is a real positive constant.

a) Show that the excitation spectrum of the above Hamiltonian has the form $E_{\mathbf{k}} = (\xi^2_{\mathbf{k}} + \Delta^2_{\mathbf{k}})^{1/2}$. Find $\Delta_{\mathbf{k}}$ and verify that it exhibits d-wave symmetry. Also consider the case $\Delta_{ij} = +\Delta_0$ for all bonds. What is the symmetry in this case?

b) Show that in the d-wave case there are 4 points on the Fermi surface where the gap vanishes and consequently $E_{\mathbf{k}} = 0$. Find the location of these nodes and make a sketch, assuming the underlying Fermi surface has 1 electron per site, i.e. at half filling.

c) Near these nodes the spectrum exhibits an approximate Dirac form

$$E_{\mathbf{k}} = \sqrt{(v_F k_1)^2 + (v_\Delta k_2)^2}$$

where k_1 and k_2 are momentum components perpendicular and parallel to the Fermi surface, respectively, measured relative to the nodal point. Find velocities v_F and v_Δ in terms of t and Δ.

d) Using the above Dirac spectrum find the density of states and deduce the temperature dependence of the specific heat at low T for a d-wave superconductor.