1. (20 points) Ground state of weakly interacting bosons.
a) The Bogoliubov transformation can be expressed as

$$\begin{pmatrix} \alpha_{\mathbf{k}} \\ \alpha^\dagger_{-\mathbf{k}} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}} \\ -v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ a^\dagger_{-\mathbf{k}} \end{pmatrix},$$

with $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ real even functions of $\mathbf{k}$. Show that, with the condition $u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2 = 1$, the matrix $U_{\mathbf{k}}$ in the above equation describes a symplectic (not unitary) transformation. Consult your favorite linear algebra book or Wikipedia for the definition of a symplectic matrix.

b) Find factors $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ required to bring the Bogoliubov Hamiltonian into the diagonal form $H = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}}$. To this end use substitution $u_{\mathbf{k}} = \cosh\theta_{\mathbf{k}}$ and $v_{\mathbf{k}} = \sinh\theta_{\mathbf{k}}$ which automatically assures $u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2 = 1$. Then show that $\hbar\omega_{\mathbf{k}} = \sqrt{\hbar^2\Omega_{\mathbf{k}}^2 - \eta_{\mathbf{k}}^2}$ and

$$\tanh 2\theta_{\mathbf{k}} = -\frac{NV_{\mathbf{k}}}{\epsilon_{\mathbf{k}} + NV_{\mathbf{k}}}.$$

c) The ground state $|\Phi_0\rangle$ has the property

$$\alpha_{\mathbf{k}}|\Phi_0\rangle = 0, \quad \text{for all } \mathbf{k}. \tag{1}$$

Explain why this is so. Then show that the average number of bosons occupying a state with momentum $\mathbf{k}$ in the ground state is

$$\bar{n}_{\mathbf{k}} \equiv \langle\Phi_0|a^\dagger_{\mathbf{k}} a_{\mathbf{k}}|\Phi_0\rangle = v_{\mathbf{k}}^2.$$

Assuming constant $V_{\mathbf{k}}$ sketch $\bar{n}_{\mathbf{k}}$ versus $|\mathbf{k}|$ and comment on the result.

d) Find the explicit expression for the ground state wavefunction $|\Phi_0\rangle$. This should be in a form $\hat{O}|0\rangle$ where $\hat{O}$ is an operator composed of $a^\dagger_{\mathbf{k}}$'s and $|0\rangle$ is the vacuum state (no bosons). *Hint:* Note that Eq. (1) implies $a_{\mathbf{k}}|\Phi_0\rangle = (v_{\mathbf{k}}/u_{\mathbf{k}})a^\dagger_{-\mathbf{k}}|\Phi_0\rangle$. Consider $\hat{O}$ of the form $\Pi_{\mathbf{k}} \exp(z_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}})$; also note that $[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$.

2. (10 points) In the Bogoliubov model for liquid ^{4}He calculate the temperature dependence of the condensate density $n_0 = N_0/\Omega$ at low T. Consider separately cases of short range ($V_{\mathbf{q}} = V_0 =$const) and long range ($V_{\mathbf{q}} = e^2/q^2$) interactions.

Hints: It is easiest to calculate the T dependence of the non-condensate density $n'(T)$ first and then deduce $n_0(T)$ from the requirement that

$$n_0(T) + n'(T) = n,$$

with $n = N/\Omega$ the total density which remains constant. Write $n'(T) = \Omega^{-1} \sum_{\mathbf{k}} \langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle$, then express it in terms of $\alpha_{\mathbf{k}}$ operators and use the fact that the Hamiltonian is diagonal in these to evaluate the average. Assume that temperature is low enough so that $\mu \approx 0$. Extract the leading T-dependence of $n'(T) - n'(0)$ by neglecting all temperature dependence of the integrand except that coming from the Bose occupation factor and using the small k approximation for $\omega_{\mathbf{k}}$. Once you have the result check that this is in fact a reasonable approximation.

3. (20 points) Consider a simple cubic lattice with spacing a in dimension d, comprised of ions of mass M interacting via nearest neighbor harmonic potential with spring constant K. Answer the questions (a-d) below for $d = 1, 2, 3$ (in most cases a simple answer in terms of d emerges, e.g. T^{d-1}).

a) Calculate the exact phonon spectrum $\omega_{\mathbf{k}\mu}$ and verify that for long wavelengths one obtains $\omega_{\mathbf{k}\mu} \simeq c|\mathbf{k}|$. Determine the sound velocity c.

b) In the above long wavelength limit calculate the phonon density of states $D(\omega)$ and deduce the T-dependence of the specific heat at low temperatures.

c) Propose the form of Debye approximation for this crystal suitable for various dimensions and determine Θ_D in terms of M, K and particle number N.

d) According to the Lindemann criterion the lattice melts when the root-mean-square fluctuation of the ion position reaches a significant fraction of the lattice constant: $\sqrt{\langle \mathbf{u}^2 \rangle} = c_L a$, where $c_L = 0.10 - 0.15$ is the Lindemann number. Calculate the T-dependence of $\langle \mathbf{u}^2 \rangle$ for our crystal and determine its melting temperature T_M. What does this imply for the stability of low-dimensional crystals?

e) Polonium metal has cubic structure with lattice constant $a = 3.34$Å, atomic mass number 209 and melting point $T_M = 527$K. Based on this information estimate its spring constant K, Debye temperature Θ_D, sound velocity c (all in physical units) and comment on these results. *Note:* Isotopes of Polonium are rare, radioactive, and dangerous to human health. Don't try to verify these results at home!