P502	Problem set # 1	Due: Oct. 4, 14:30

1. (10 points) In the first quantized notation the operator for electron spin density is $\mathbf{S}(\mathbf{r}) = \frac{\hbar}{2}\vec{\sigma}\sum_i \delta(\mathbf{r}-\mathbf{r}_i)$ with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ a vector of Pauli matrices.

a) Construct the spin operator $\hat{\mathbf{S}}(\mathbf{r})$ in second quantization in terms of electron field operators $\hat{\psi}_\lambda(\mathbf{r})$ and electron creation/annihilation operators c, $c^\dagger$. Use the plane wave basis of single particle states.

b) Show that the operator of total spin, $\mathbf{S}_{\text{tot}} = \int d^3r \hat{\mathbf{S}}(\mathbf{r})$ commutes with the Hamiltonian of electrons interacting via Coulomb forces discussed in class. *Hint:* This can be done using the field operator representation; i.e. compute the commutator $[\hat{\mathbf{S}}(\mathbf{r}), \hat{\psi}_\lambda(\mathbf{r}')]$ and use the result to find $[\mathbf{S}_{\text{tot}}, \hat{H}]$, or directly with $c_{k\alpha}$ operators.

2. (20 points) Stoner Instability. Consider a polarized electron gas in which $N_\pm$ denotes the number of electrons with spin up and down. Electrons are assumed to interact via the usual Coulomb interaction.

a) Find the ground-state energy to first order in the interaction potential as a function of $N = N_+ + N_-$ and the polarization $M = (N_+ - N_-)/N$.

b) Prove that ferromagnetic state ($M = 1$) has a lower energy than the unmagnetized state ($M = 0$) if r_s exceeds critical value r_s^c. Explain why this is so. Find r_s^c.

c) Show that $\partial^2(E/N)/\partial M^2|_{M=0}$ becomes negative for $r_s > (3\pi^2/2)^{\frac{2}{3}} \simeq 6.03$. Discuss the physical significance of the two critical densities. What happens for $r_s^c < r_s < 6.03$?

3. (15 points) Consider single electron excitation spectrum in the Hartree-Fock approximation derived in class.

a) Show that the Hartree-Fock correction leads to divergent electron velocity $v_k = \partial\epsilon_k/\partial k$ at the fermi surface. Determine the degree of this divergence.

b) Study the density of states $\rho(\epsilon)$ and show that the Hartree-Fock correction makes it vanish as the fermi surface is approached. What does this imply for the validity of the Sommerfeld expansion for calculation of thermodynamic quantities such as the specific heat?

c) Calculate the Hartree-Fock correction to the electron energies for the *screened* Coulomb (Yukawa) potential $V(r) \sim e^{-\mu r}/r$, and show that this immediately corrects the unphysical behavior found in parts (a) and (b).