Residual Resistivity and Localization in Disordered Materials

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I show how to recover the Drude formula for the conductivity of a normal metal (with the correct transport time) by summing the series of ladder diagrams in the current-current correlation function. Then I discuss how to implement the quantum corrections associated with maximally crossed diagrams. It turns out that, for dimension $d \leq 2$, these corrections are divergent, signalling the onset of localization.

I. INTRODUCTION

Even in the limit of zero temperature, in which the inelastic scattering of electrons by phonons or other excitations is suppressed, the resistance of normal metals is found to be finite. This residual resistance is due to the presence of randomly distributed impurities in any realistic material. Although the particular distribution of the impurities is unknown, a great deal of information can be extracted from disorder-averaged quantities. These, in fact, are expected to describe the system very accurately in the thermodynamic limit, which in principle contains all the realizations of disorder (in the case of "self-averaging").

Along the course, we have developed a formalism to calculate many-particle Green's functions by conveniently organizing perturbative expansions in the form of Feynman diagrams. In special, one can express linear response properties of a system in terms of two-particle Green's functions. The idea of the first part of this project is to apply diagrammatics to recover the classical expression for the conductivity of a normal metal – known as Drude formula – when weak scattering off impurities causes the current relaxation. In order to do so, we have to sum an infinite series of so-called ladder diagrams [1]. The second part of the project focuses on the contribution of the previously neglected crossed diagrams. I will comment on how Anderson's localization – the fact that the electron's wave function becomes localized in a random potential [2] – appears in this formalism.

II. APPROXIMATION FOR THE ONE-PARTICLE GREEN'S FUNCTION

Let us briefly review some of the results for Green's functions in disordered systems. We shall keep the same notation as in the lecture notes. The effect of impurities is represented by a disorder potential $V(\vec{r})$ in the Hamiltonian. As a starting point, we are interested in the disorder-averaged one-particle Green's function, denoted by $\bar{G}(\vec{r}_f, t_f; \vec{r}_i, t_i) = \langle G(\vec{r}_f, t_f; \vec{r}_i, t_i) \rangle_{dis}$. We assume Gaussian disorder, such that $\langle V(\vec{r}) \rangle_{dis} = 0$ and $\langle V(\vec{r_1}) V(\vec{r_2}) \rangle_{dis} = W(\vec{r_1} - \vec{r_2})$. Moreover, all correlations involving an odd number of V's vanish and the ones involving even numbers can be factorized as products of two-point correlation functions. Then the perturbative expansion for $\bar{G}(\vec{r}_f, t_f; \vec{r}_i, t_i)$ contains only diagrams of even order in $V(\vec{r})$. The bare Green's function $G_0(\vec{r}_f, t_f; \vec{r}_i, t_i)$ is represented by a single straight line, while the impurities correspond to points connected by dashed lines in all possible ways (Figure 1). A dashed line connecting two points $\vec{r}$ and $\vec{r}'$ stands for $W(\vec{r} - \vec{r}')$. As usual, it is more interesting to work in momentum space.

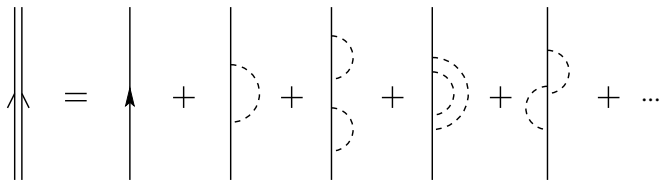

Figure 1: Diagrammatic expansion for the disorder-averaged one particle Green's function.

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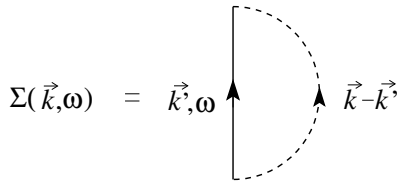

Figure 2: Proper self-energy in the Born approximation.

However, before taking the average over disorder, the system lacks translation invariance and we have to define

$$\mathcal{G}(\vec{k},\vec{p},i\omega_n)=\int d\vec{r}\int d\vec{r}'\int_0^{\beta\hbar}d\tau\, e^{-i\vec{k}\cdot\vec{r}}e^{+i\vec{p}\cdot\vec{r}'}e^{i\omega_n\tau}\mathcal{G}(\vec{r},\tau;\vec{r}',0) \tag{1}$$

as the propagator associated with the probability of going in with momentum $\vec{p}$ and coming out with momentum $\vec{k}$. Here, τ is the imaginary time and we work with fermionic Matsubara frequencies $\omega_n=(2n+1)\,\pi/\beta\hbar$, where $\beta=(k_BT)^{-1}$. After averaging over disorder, we find that we can associate momentum to the dashed lines and replace them by

$$W\left(\vec{q}\right)=\int d\vec{r}\,e^{-i\vec{q}\cdot\vec{r}}\,W\left(\vec{r}\right). \tag{2}$$

Since the momentum transferred to a dashed line at one point is returned at some other point, we recover momentum conservation in the form $\left\langle \mathcal{G}(\vec{k},\vec{p},i\omega_n)\right\rangle_{dis}=\delta_{\vec{k}\vec{p}}\bar{\mathcal{G}}(\vec{k},i\omega_n)$.

The most immediate effect of impurity scattering is to introduce a finite lifetime for the quasiparticles. For our purposes, it is enough to calculate the proper self-energy in the Born approximation, given by the diagram in Figure 2. This yields

$$\Sigma(\vec{k},i\omega_n)\approx\frac{1}{\hbar^2}\frac{1}{V}\sum_{\vec{k}'}\mathcal{G}_0(\vec{k}',i\omega_n)W(\vec{k}-\vec{k}'), \tag{3}$$

where $\mathcal{G}_0(\vec{k},i\omega_n)$ is the temperature Green's function for free particles

$$\mathcal{G}_0(\vec{k},i\omega_n)=\frac{1}{i\omega_n-\epsilon_k/\hbar}, \tag{4}$$

with dispersion $\epsilon_k=\hbar^2k^2/2m$. At low energies, we pick up only the contribution from momenta close to the Fermi surface. In this limit, and assuming spherical symmetry, $W(\vec{k}-\vec{k}')$ will depend only on the angle θ between $\vec{k}$ and $\vec{k}'$. Thus we get from Eq. (3)

$$\Sigma(\vec{k},i\omega_n)\approx\frac{mk_F}{(2\pi)^3\hbar^3}\int d\Omega\,W\left(\theta\right)\int_{-\Lambda}^{+\Lambda}d\xi_{k'}\frac{1}{i\omega_n-\xi_{k'}}=-\frac{i}{2\tau_c}\,\mathrm{sgn}\left(\omega_n\right), \tag{5}$$

where $\xi_k=v_F\left(k-k_F\right)$, Λ is an ultraviolet cutoff and we have taken the limit $\Lambda/\left|\omega_n\right|\to\infty$. We have also defined the mean time between collisions τ_c by

$$\frac{1}{\tau_c}=\frac{mk_F}{4\pi^2\hbar^3}\int d\Omega\,W\left(\theta\right). \tag{6}$$

Furthermore, we can define the mean free path by $\ell=v_F\tau_c$. The disorder-averaged Green's function in the Born approximation is

$$\bar{\mathcal{G}}(\vec{k},i\omega_n)=\frac{1}{\mathcal{G}_0^{-1}(\vec{k},i\omega_n)-\Sigma(\vec{k},i\omega_n)}\approx\frac{1}{i\omega_n-\epsilon_k/\hbar+(i/2\tau_c)\ \mathrm{sgn}\left(\omega_n\right)}. \tag{7}$$

III. DERIVATION OF THE DRUDE FORMULA FOR CONDUCTIVITY

Let us now turn to the calculation of the electrical conductivity. We should be able to write the current induced by an external electric field in the form

$$J_\alpha(\vec{r},t) = \int d\vec{r}\,' \int dt'\, \sigma_{\alpha\beta}(\vec{r},\vec{r}\,',t-t')\, E_\beta(\vec{r}\,',t'), \tag{8}$$

where $\sigma_{\alpha\beta}(\vec{q},\omega)$ is the conductivity tensor with indices $\alpha,\beta = x,y,z$. If the system is isotropic, we must have $\sigma_{\alpha\beta} = \sigma\delta_{\alpha\beta}$. For a harmonic field $\vec{E}(\vec{r}\,',t') = \vec{E}_0 e^{i\vec{q}\cdot\vec{r}\,'}e^{-i\omega t'}$ in a homogeneous system, this simplifies to $J_\alpha(\vec{r},t) = \sigma_{\alpha\beta}(\vec{q},\omega)\,E_{0\beta}e^{i\vec{q}\cdot\vec{r}}e^{-i\omega t}$. It is a result of linear response theory that $\sigma_{\alpha\beta}(\vec{q},\omega)$ can be expressed in terms of the current-current correlation function

$$\sigma_{\alpha\beta}(\vec{q},\omega) = \frac{i}{\omega}\left[\Pi^R_{\alpha\beta}(\vec{q},\omega) + \frac{ne^2}{m}\delta_{\alpha\beta}\right], \tag{9}$$

where n is the electron density and $\Pi^R_{\alpha\beta}(\vec{q},\omega)$ is the retarded correlation function

$$\Pi^R_{\alpha\beta}(\vec{q},\omega) = -\frac{i}{\hbar V}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\theta(t)\left\langle\left[j_\alpha(\vec{q},t), j_\beta(-\vec{q},0)\right]\right\rangle, \tag{10}$$

This is called the Kubo formula and a derivation can be found in Ref. [3]. In order to get the DC conductivity, one takes $\vec{q}=0$ first and then the limit $\omega\to 0$. Here, $\vec{j}(\vec{q})$ is the Fourier transform of the current operator $\vec{j}(\vec{r}) = (e/2m)\sum_i\left[\vec{p}_i\delta(\vec{r}-\vec{r}_i) + \delta(\vec{r}-\vec{r}_i)\,\vec{p}_i\right]$, which in second quantization becomes

$$\vec{j}(\vec{r}) = -\frac{ie\hbar}{2m}\left[\psi^\dagger(\vec{r})\,\nabla\psi(\vec{r}) - \nabla\psi^\dagger(\vec{r})\,\psi(\vec{r})\right], \tag{11}$$

where $\psi(\vec{r})$ is the field operator (we can forget the spin indices here). Then we have

$$j_\alpha(\vec{q}) = \int d\vec{r}\, e^{-i\vec{q}\cdot\vec{r}} j_\alpha(\vec{r}) = \frac{e\hbar}{m}\sum_{\vec{k}} k_\alpha c^\dagger_{\vec{k}-\frac{\vec{q}}{2}} c_{\vec{k}+\frac{\vec{q}}{2}}. \tag{12}$$

Note, though, that $\Pi^R_{\alpha\beta}$ in Eq. (10), with only one $\vec{q}$, is valid when we have translational invariance. When momentum is not conserved, we can still write $J_\alpha(\vec{r},t)\propto E_{0\beta}e^{i\vec{q}\cdot\vec{r}}e^{-i\omega t}$, but with $\sigma_{\alpha\beta}(\vec{q},\omega)$ replaced by the generalization

$$\sigma_{\alpha\beta}(\vec{q},\omega) \to \frac{i}{\omega}\left[\sum_{\vec{q}\,'} e^{i(\vec{q}\,'-\vec{q})\cdot\vec{r}}\Pi^R_{\alpha\beta}(\vec{q}\,',\vec{q},\omega) + \frac{ne^2}{m}\delta_{\alpha\beta}\right], \tag{13}$$

where

$$\Pi^R_{\alpha\beta}(\vec{q}\,',\vec{q},\omega) = -\frac{i}{\hbar V}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\theta(t)\left\langle\left[j_\alpha(\vec{q}\,',t), j_\beta(-\vec{q},0)\right]\right\rangle. \tag{14}$$

It is easier to obtain the retarded correlation function from the temperature correlation function

$$\Pi_{\alpha\beta}(\vec{q}\,',\vec{q},i\omega_m) = -\frac{1}{\hbar V}\int_0^{\beta\hbar} d\tau\, e^{i\omega_m\tau}\left\langle T_\tau j_\alpha(\vec{q}\,',\tau)\, j_\beta(-\vec{q},0)\right\rangle, \tag{15}$$

where ω_m are bosonic Matsubara frequencies $\omega_m = 2\pi m/\beta\hbar$, by taking the analytical continuation $i\omega_m\to\omega+i\eta$. Using Eq. (12), we get

$$\begin{aligned}\Pi_{\alpha\beta}(\vec{q}\,',\vec{q},i\omega_m) &= -\left(\frac{e\hbar}{m}\right)^2 \lim_{\eta\to 0^+}\frac{1}{\hbar V}\sum_{\vec{k}\,\vec{k}\,'} k'_\alpha k_\beta \int_0^{\beta\hbar} d\tau\, e^{i\omega_m\tau}\left\langle T_\tau c^\dagger_{\vec{k}\,'-\frac{\vec{q}\,'}{2}}(\tau+\eta)\, c_{\vec{k}\,'+\frac{\vec{q}\,'}{2}}(\tau)\, c^\dagger_{\vec{k}+\frac{\vec{q}}{2}}(\eta)\, c_{\vec{k}-\frac{\vec{q}}{2}}(0)\right\rangle \\ &= \left(\frac{e}{m}\right)^2\frac{\hbar}{V}\sum_{\vec{k}\,\vec{k}\,'} k'_\alpha k_\beta \frac{1}{\beta\hbar}\sum_n \mathcal{G}\left(\vec{k}\,'+\frac{\vec{q}\,'}{2},\vec{k}+\frac{\vec{q}}{2},i\omega_n+i\omega_m\right)\mathcal{G}\left(\vec{k}-\frac{\vec{q}}{2},\vec{k}\,'-\frac{\vec{q}\,'}{2},i\omega_n\right).\end{aligned} \tag{16}$$

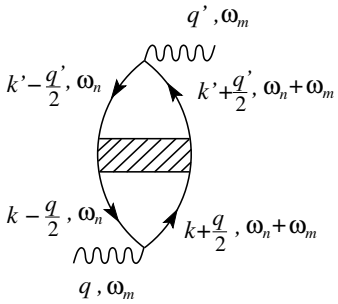

Figure 3: Bubble diagram without conservation of momentum.

Therefore, $\Pi_{\alpha\beta}\left(\vec{q}',\vec{q},i\omega_m\right)$ is a two-particle correlation function which corresponds to the polarization bubble in Figure 3. In this diagram, one goes in with momentum $\vec{q}$ and frequency ω_m and comes out with the same frequency, but a different momentum $\vec{q}'$. However, we are really interested in the disorder-averaged correlation function. As we have discussed, this procedure amounts to drawing diagrams with the impurity points connected by "interaction" lines in all possible ways. Some of these diagrams are represented in Figure 4. Note that our zoo of diagrams is richer than if we just replaced the one-particle Green's function by $\bar{\mathcal{G}}(\vec{k},i\omega_n)$ shown in Figure 1 because we can contract a potential $V\left(\vec{r}\right)$ that comes from the particle line with another $V\left(\vec{r}'\right)$ that comes from the hole line. This leads to diagrams like the ones in Figure 4-c, 4-e and 4-f. Besides, when we average over disorder, all the momenta carried by the interaction lines remain inside the bubble. As a consequence, the momentum that leaves the bubble equals the momentum that enters it. In other words, we recover translational invariance in the form

$$\left\langle \Pi_{\alpha\beta}\left(\vec{q}',\vec{q},i\omega_m\right)\right\rangle_{dis} = \overline{\Pi}_{\alpha\beta}\left(\vec{q},i\omega_m\right)\delta_{\vec{q}\vec{q}'}. \tag{17}$$

Let us sort the diagrams in Figure 4. Diagrams such as 4-a, 4-b and 4-d are obtained by dressing the one-particle Green's functions with the interaction lines. If we just replace $\mathcal{G}_0(\vec{k},i\omega_n)$ by $\bar{\mathcal{G}}(\vec{k},i\omega_n)$ in Eq. (7), diagrams 4-a and 4-b are automatically included (as well as all the diagrams with particle or hole lines generated by the self-energy in the Born approximation). The rainbow diagram 4-d is not included in our approximation for $\bar{\mathcal{G}}(\vec{k},i\omega_n)$, but including this kind of diagram only renormalizes the time τ_c. Diagrams such as 4-c and 4-e represent scattering between the particle and the hole with no crossed interaction lines. These are called *ladder diagrams*, and it turns out we can sum their entire series exactly. We shall do that soon, but first we have to argue why we can discard crossed diagrams like 4-f. Figure 5 shows a closer look at this diagram with the momentum carried by the fermion and interaction lines, as given by the condition of conservation of momentum at the vertices. At low energies, conservation of energy introduces the constraint that the fermionic momenta lie on the Fermi surface. For small $\vec{q}$ and $|\vec{k}| = k_F$, it is easy to make sure that $\vec{k}+\vec{q}_1$ and $\vec{k}+\vec{q}_2$ are on the Fermi surface. However, the extra condition that so does $\vec{k}+\vec{q}_1+\vec{q}_2$ poses a serious restriction in the integration over $\vec{q}_1$ and $\vec{q}_2$. Since the allowed energy range here is of order $\hbar/\tau_c$ around the Fermi surface, the contribution of this diagram will be smaller than the one of 4-e (for which there is no such restriction) by a factor of $\left(k_F\ell\right)^{-1} \ll 1$ in the weak-scattering limit (defined by $\tau_c \gg \hbar/\epsilon_F$, with ϵ_F the Fermi energy). So it seems that we can just forget about crossed diagrams and go for the ladder diagrams to calculate the DC conductivity. To tell you the truth, this is quite a naive argument because it focuses on the amplitude of these processes, but says nothing about the ω-dependence. We shall come back to this subject when we discuss localization.

Now we work out the expansion of $\overline{\Pi}_{\alpha\beta}\left(\vec{q},i\omega_m\right)$ containing only ladder diagrams. According to Eq. (16), this is

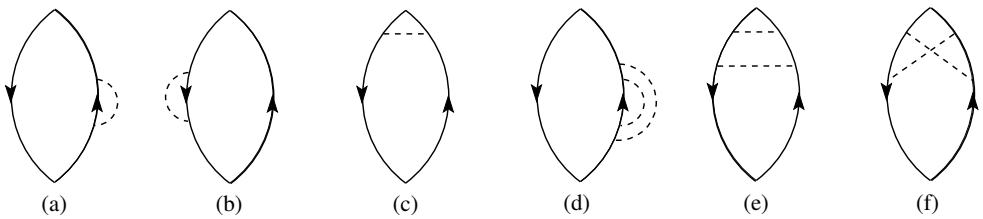

Figure 4: Possible diagrams for the two-particle correlation function at first order in $W\left(\vec{q}\right)$ (a-c) and some of the diagrams at second order (d-f).

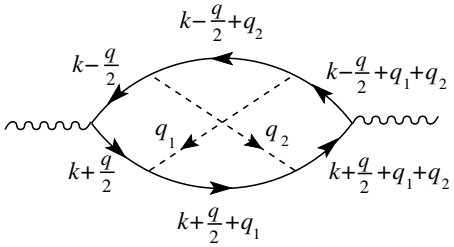

Figure 5: Crossed diagram at second order.

given by

$$\overline{\Pi}_{\alpha\beta}\left(\vec{q},i\omega_m\right)=\left(\frac{e}{m}\right)^2\frac{\hbar}{V}\sum_{\vec{k}\,\vec{k}\,'}k'_\alpha k_\beta\,\frac{1}{\beta\hbar}\sum_n\left\langle\mathcal{G}(\vec{k}'_+,\vec{k}_+)\mathcal{G}(\vec{k}_-,\vec{k}'_-)\right\rangle_{dis}^{ladder}, \tag{18}$$

where we have adopted the shorthand notation $\mathcal{G}(\vec{k}'_+,\vec{k}_+)=\mathcal{G}(\vec{k}\,'+\frac{\vec{q}'}{2},\vec{k}+\frac{\vec{q}}{2},i\omega_n+i\omega_m)$ and $\mathcal{G}(\vec{k}'_-,\vec{k}_-)=\mathcal{G}(\vec{k}-\frac{\vec{q}}{2},\vec{k}\,'-\frac{\vec{q}'}{2},i\omega_n)$. By $\langle\,\rangle_{dis}^{ladder}$, we mean the infinite series of diagrams represented in Figure 6 with the bare propagator replaced by $\bar{\mathcal{G}}(\vec{k},i\omega_n)$ (note that this is the same as the bubble with open vertices). We can try to calculate $\overline{\Pi}_{\alpha\beta}(\vec{q},i\omega_m)$ by summing over $\vec{k}$ first. We define

$$\overline{\Pi}_\beta(\vec{k}'_+,\vec{k}'_-)=\sum_{\vec{k}}k_\beta\left\langle\mathcal{G}(\vec{k}'_+,\vec{k}_+)\mathcal{G}(\vec{k}_-,\vec{k}'_-)\right\rangle_{dis}^{ladder}. \tag{19}$$

Then, according to Figure 6, this is given by the expansion

$$\begin{aligned}\overline{\Pi}_\beta(\vec{k}'_+,\vec{k}'_-) &= \bar{\mathcal{G}}(\vec{k}'_+)\bar{\mathcal{G}}(\vec{k}'_-)\sum_{\vec{k}}k_\beta\left[\delta_{\vec{k}\vec{k}\,'}+\frac{1}{\hbar^2V}\sum_{\vec{q}_1}W\left(q_1\right)\bar{\mathcal{G}}(\vec{k}'_+\!+\vec{q}_1)\bar{\mathcal{G}}(\vec{k}'_-\!+\vec{q}_1)\delta_{\vec{k},\vec{k}\,'+\vec{q}_1}+\right.\\ &\left.+\frac{1}{\hbar^4V^2}\sum_{\vec{q}_1\vec{q}_2}W\left(q_1\right)\bar{\mathcal{G}}(\vec{k}'_+\!+\vec{q}_1)\bar{\mathcal{G}}(\vec{k}'_-\!+\vec{q}_1)W\left(q_2\right)\bar{\mathcal{G}}(\vec{k}'_+\!+\vec{q}_1+\vec{q}_2)\bar{\mathcal{G}}(\vec{k}'_-\!+\vec{q}_1+\vec{q}_2)\delta_{\vec{k},\vec{k}\,'+\vec{q}_1+\vec{q}_2}+...\right]\\ &= \bar{\mathcal{G}}(\vec{k}'_+)\bar{\mathcal{G}}(\vec{k}'_-)\left[k'_\beta+\frac{1}{\hbar^2V}\sum_{\vec{p}}p_\beta W(\vec{p}-\vec{k}')\bar{\mathcal{G}}(\vec{p}_+)\bar{\mathcal{G}}(\vec{p}_-)+\right.\\ &\left.+\frac{1}{\hbar^4V^2}\sum_{\vec{p}\vec{p}\,'}p'_\beta W(\vec{p}-\vec{k}')\bar{\mathcal{G}}(\vec{p}_+)\bar{\mathcal{G}}(\vec{p}_-)W\left(\vec{p}\,'-\vec{p}\right)\bar{\mathcal{G}}(\vec{p}'_+)\bar{\mathcal{G}}(\vec{p}'_-)+...\right],\end{aligned} \tag{20}$$

where we have shifted the sums by setting $\vec{p}=\vec{q}_1+\vec{k}\,'$, $\vec{p}\,'=\vec{q}_2+\vec{p}$, etc. Then we recognize that Eq. (20) has the form of a Dyson's equation for $\overline{\Pi}_\beta(\vec{k}'_+,\vec{k}'_-)$

$$\overline{\Pi}_\beta(\vec{k}'_+,\vec{k}'_-)=\bar{\mathcal{G}}(\vec{k}'_+)\bar{\mathcal{G}}(\vec{k}'_-)\left[k'_\beta+\frac{1}{\hbar^2V}\sum_{\vec{p}}W(\vec{p}-\vec{k}')\overline{\Pi}_\beta\left(\vec{p}_+,\vec{p}_-\right)\right]. \tag{21}$$

All terms in the expansion of Eq. (20) include vector components in the β direction. For this reason, we expect the sum in Eq. (21) to be proportional to k_β as well. So let us assume the ansatz

$$\frac{1}{\hbar^2V}\sum_{\vec{p}}W(\vec{p}-\vec{k}')\overline{\Pi}_\beta\left(\vec{p}_+,\vec{p}_-\right)=k'_\beta\Lambda\left(\omega_m,\omega_n\right), \tag{22}$$

where $\Lambda\left(\omega_m,\omega_n\right)$ must be independent of $|\vec{k}\,'|$ in the low-energy limit (all vectors on the Fermi surface). Then, if we multiply Eq. (21) by $W(\vec{k}\,'-\vec{p}\,')$ and sum over $\vec{k}\,'$, we get

$$p'_\beta\Lambda\left(\omega_m,\omega_n\right)=\frac{1}{\hbar^2V}\sum_{\vec{k}\,'}W(\vec{k}\,'-\vec{p}\,')\overline{\Pi}_\beta(\vec{k}'_+,\vec{k}'_-)=\frac{1}{\hbar^2V}\sum_{\vec{k}\,'}W(\vec{k}\,'-\vec{p}\,')\bar{\mathcal{G}}(\vec{k}'_+)\bar{\mathcal{G}}(\vec{k}'_-)k'_\beta\left[1+\Lambda\left(\omega_m,\omega_n\right)\right], \tag{23}$$

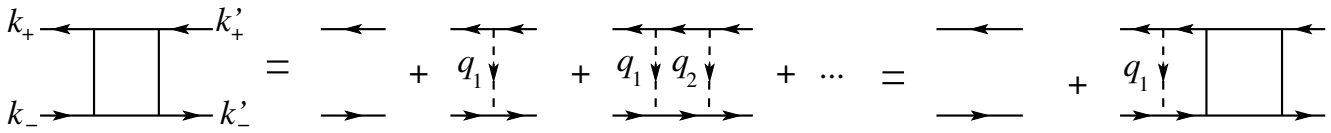

Figure 6: Ladder diagrams.

which can be rewritten (in the thermodynamic limit) as

$$\frac{\Lambda\left(\omega_m,\omega_n\right)}{1+\Lambda\left(\omega_m,\omega_n\right)}\vec{p}\,' = \frac{1}{\hbar^2}\int\frac{d\vec{k}}{\left(2\pi\right)^3}W(\vec{k}-\vec{p}\,')\bar{\mathcal{G}}(\vec{k}_+)\bar{\mathcal{G}}(\vec{k}_-)\vec{k}. \tag{24}$$

Since we assume that $W = W\left(\theta\right)$ (with θ the angle between $\vec{k}$ and $\vec{p}\,'$), the only component of $\vec{k}$ that survives the integration is the one parallel to $\vec{p}\,'$. Therefore, we can replace $\vec{k}$ in the integrand by $\approx k_F\cos\theta\,\hat{\vec{p}}\,'$, where $\hat{\vec{p}}\,'$ stands for the the unit vector in the direction of $\vec{p}\,'$. Let us set $\vec{q}=0$ at once (then $\vec{k}_+ = \vec{k}_-$) to simplify the calculations (remember, we are interested in the DC conductivity). By substituting the expressions for the one-particle Green's function in Eq. (7), we obtain

$$\frac{\Lambda}{1+\Lambda} = \frac{mk_F}{\left(2\pi\right)^3\hbar^3}\int d\Omega\, W\left(\theta\right)\cos\theta\int_{-\infty}^{+\infty}d\xi_k\,\frac{1}{i\left(\omega_n+\omega_m\right)-\xi_k+\frac{i}{2\tau_c}\mathrm{sgn}\left(\omega_n+\omega_m\right)}\,\frac{1}{i\omega_n-\xi_k+\frac{i}{2\tau_c}\mathrm{sgn}\left(\omega_n\right)}. \tag{25}$$

The only possibility of a non-vanishing integral is that the poles of the integrand are in opposite sides of the real axis. If we choose the external frequency ω_m to be positive, we must have $\omega_n<0$ and $|\omega_n|<\omega_m$. After integrating over ξ_k and some manipulation, we are left with

$$\Lambda\left(\omega_m,\omega_n\right) = \frac{i/\tau_1}{i\omega+i/\tau_{tr}}\theta\left(-\omega_n\right)\theta\left(\omega_m+\omega_n\right), \tag{26}$$

where we have defined the transport time

$$\frac{1}{\tau_{tr}} = \frac{mk_F}{4\pi^2\hbar^3}\int d\Omega\, W\left(\theta\right)\left(1-\cos\theta\right), \tag{27}$$

and also $\tau_1^{-1}=\tau_c^{-1}-\tau_{tr}^{-1}$. Once we know $\Lambda\left(\omega_m,\omega_n\right)$, we get $\overline{\Pi}_\beta(\vec{k}\,'_+,\vec{k}\,'_-)$ from Eq. (21). Then, if we put this back in Eq. (18) and do the integrals over $\xi_{k'}$ and ω_n, we obtain

$$\overline{\Pi}_{\alpha\beta}\left(\vec{q}=0,i\omega\right) = \delta_{\alpha\beta}\,\frac{2e^2k_F^3}{3m\left(2\pi\right)^2}\,\frac{i\omega}{i\omega+i/\tau_{tr}}.$$

We get the retarded correlation function by taking $i\omega\to\omega+i\eta$. Substituting in Eq. (9) , we find

$$\mathrm{Re}\,\sigma\left(\omega\right) = \frac{ne^2}{m}\,\frac{\tau_{tr}}{1+\left(\omega\tau_{tr}\right)^2}. \tag{28}$$

In the limit $\omega\to 0$, we find the DC conductivity

$$\mathrm{Re}\,\sigma_0 = \frac{ne^2\tau_{tr}}{m}. \tag{29}$$

The result in Eq. (28) is the Drude formula for the conductivity of a normal metal. It is just the statement that the metal obeys Ohm's law, i.e., it has a finite, constant conductivity. Remarkably, it is the same result as derived from the Boltzmann transport equation on a semiclassical basis [3]. We can identify $Q\left(\theta\right)\equiv mk_FW\left(\theta\right)/4\pi^2\hbar^3$ as the probability of transition between two momentum states, as used in that approach. Note that the relevant time for transport properties is not the time between collisions, but the transport time τ_{tr} because only decay processes that change the direction of motion of the electrons significantly are effective in causing current relaxation. This is expressed by the factor $(1-\cos\theta)$ that appears in Eq. (27) and favors scattering with large angles. If we had neglected the ladder diagrams and calculated σ just by dressing the one-particle Green's function, we would have found that σ is proportional to τ_c instead of τ_{tr}. This comes about because the ladder diagrams are associated with processes in which particle and hole are scattered by the same impurity and can then recombine, restoring the current. The equivalent term in Boltzmann equation is the one that accounts for transitions into a given momentum state, as opposed to the scattering out of this state due to decay. However, note that if the scattering is isotropic ($W=W_0=$ const.) the contribution of the ladder diagrams (which involves $\cos\theta$) vanishes and $\tau_{tr}=\tau_c$.

IV. QUANTUM CORRECTIONS AND LOCALIZATION

There is something beyond the classical result, though. Those are the quantum corrections associated with the crossed diagrams we have neglected. As first pointed out by Langer and Neal [4], these diagrams have a singular dependence on the impurity concentration. At each order, the most divergent contributions are given by the maximally crossed diagrams (Figure 7), and the entire series of these diagrams has to be summed together. This summation can also be done, and in fact it is very similar to summing the ladder diagrams. To see the connection, note that if we flip the hole line in a maximally crossed diagram and turn it into a particle line with opposite momentum, we obtain a ladder diagram in the *particle-particle channel* (Figure 8). For $W(\vec{k}-\vec{k}\,')=W_0=$ const., the result of summing the diagrams in Figure 7 is [5]

$$\Lambda_{\vec{k}\vec{k}'}(\vec{q},\omega)=\frac{i\tau_c^{-1}W_0}{\omega+iD_0(\vec{k}+\vec{k}\,')^2}, \tag{30}$$

where $D_0=v_F^2\tau_c/d$ is the diffusion constant in d dimensions. This is sometimes called the Cooperon propagator. A similar propagator can be found in the particle-hole channel, and that is called the diffusion propagator because it is important for the density-density correlation function (which essentially measures how the electron diffuses in the system). However, while the diffusion has poles for $\omega\to 0$ and $q\to 0$, the Cooperon in Eq. (30) has poles for $\vec{p}\approx-\vec{p}\,'$ and is thus related to backscattering processes off the impurity. By inserting $\Lambda_{\vec{k}\vec{k}'}$in the current-current correlation function, in a way similar to what we did to $\overline{\Pi}_\beta(\vec{k}'_+,\vec{k}'_-)$, one can get the quantum correction to the conductivity. The result is controlled by the behavior of the integrand at $\tilde{q}\equiv|\vec{k}+\vec{k}\,'|\to 0$. Basically, one has to deal with an integral of the form

$$\int d\tilde{q}\,\frac{\tilde{q}^{d-1}}{\omega+iD_0\tilde{q}^2},$$

which gives [6, 7]

$$\delta\sigma(\omega)=\begin{cases}-\frac{e^2}{\pi\hbar}\sqrt{\frac{D_0}{\omega}} & \text{at } d=1,\\ -\frac{e^2}{2\pi^2\hbar}\ln\left(\frac{1}{\omega\tau}\right) & \text{at } d=2,\\ \text{const.}+\frac{e^2}{2\pi^2\hbar}\sqrt{\frac{\omega}{D_0}} & \text{at } d=3.\end{cases} \tag{31}$$

The corrections in $d=1$ and $d=2$ tend to decrease σ and diverge as $\omega\to 0$. This implies that perturbation theory breaks down for $d\leq 2$ and that the free electron model is not a good starting point. Indeed, Anderson [2] showed that a strong enough disorder prevents electrons from hopping through the lattice because orbitals in neighbour sites have different energies and so do not mix significantly, even when the spatial overlap is large. The electrons become localized in the sense that the wave function decays exponentially $\psi(x)\sim e^{-x/\xi}$ with a characteristic *localization length* ξ. As a result, the conductivity must vanish. What our perturbation theory tells us is that for $d\leq 2$ localization takes place for any arbitrarily small disorder. There is an interesting interpretation of the maximally crossed diagrams as associated with the interference between wave packets moving in opposite directions along self-intersecting paths [7]. This essentially quantum effect is not revealed by semi-classical approaches. Note that the equivalence between particle and hole propagators relies on time-reversal symmetry. Once the latter is broken (by, say, an external magnetic field), the infrared divergences in the Cooperon propagator are removed.

The divergences in $\delta\sigma(\omega)$ are consistent with the results of the scaling theory of localization [8]. In this theory, one looks at how the normalized conductance $g=G/\left(e^2/\hbar\right)$, where G is the conductance, varies with the length L of the system, by analyzing the "Thouless number"

$$\beta(g)=\frac{d\ln g}{d\ln L}. \tag{32}$$

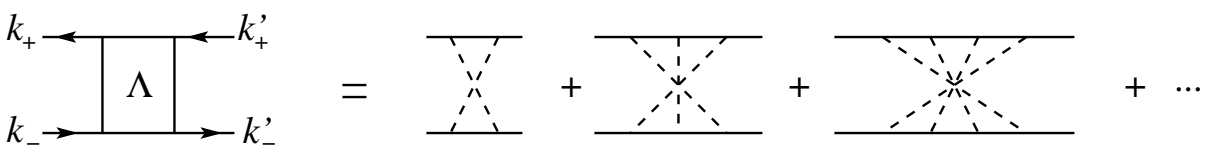

Figure 7: Maximally crossed diagrams.

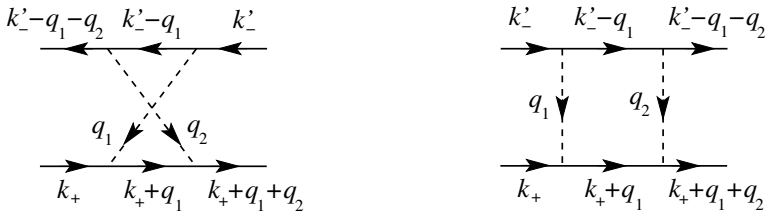

Figure 8: Maximally crossed diagrams are equivalent to ladder diagrams in the particle-particle channel.

Here, L^{-1} plays the role of setting the low-energy scale in a finite system, as does ω in the perturbative calculation in the thermodynamic limit. An important assumption, which is supported by perturbation theory, is that β is a function of $g(L)$ only. The result for weak disorder is [9]

$$\beta(g) = (d-2) - \frac{a}{g}, \tag{33}$$

where $a = \pi^{-2}$ for an electron gas. This means that the conductance will decrease with the system size for both $d = 1$ and $d = 2$. Indeed, one finds for the scale-dependent conductivity:

$$\begin{aligned} \sigma_{3D}(L) &= \sigma_0 - \frac{e^2}{\hbar\pi^3}\left(\frac{1}{\ell} - \frac{1}{L}\right), \\ \sigma_{2D}(L) &= \sigma_0 - \frac{e^2}{\hbar\pi^2}\ln\left(\frac{L}{\ell}\right), \\ \sigma_{1D}(L) &= \sigma_0 - \frac{e^2}{\hbar\pi^2}(L-\ell). \end{aligned} \tag{34}$$

So, in the limit $L \gg \ell$, the quantum correction diverges for $d \leq 2$. Note, however, that $d = 2$ is the marginal case in Eq. (33). The flow to zero conductivity is very slow (logarithmic) for this case, which means that one needs an exponentially large system ($L \sim \xi_{2D} = \ell \exp(\pi k_F \ell/2)$) to observe a reduction in the conductivity. This is known as weak localization. For $d = 3$, the correction remains finite when $L \to \infty$, which implies that the electron states are extended and the classical result is correct in the limit $k_F\ell \gg 1$. A localized phase does exist in 3D for strong disorder, but this is not seen by our weak-coupling perturbation theory. In fact, in 3D one can get a metal-insulator transition by varying the chemical potential relatively to the disorder strength. The energy where this transition occurs is called the "mobility edge". It is important to point out, though, that a metal-insulator transition is *not* a true thermodynamic phase transition, because conductivity is not an equilibrium property. Thermodynamic properties are smoothly varying functions of disorder: There are no divergences arising from crossed diagrams in the one-particle Green's function.

The problem of localization is quite vast. Many interesting issues arise, for example, when electron-electron interaction is introduced. If you are interested, you can take a look at the reviews in references [7] and [9].

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