Bosonization: Exploiting the Bosonic character of Fermionic systems

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Abstract

We present a detailed derivation of the representation of one-dimensional Fermionic operators in terms of Bosonic operators, the so-called Bosonization identity. This identity is independent of any specific Hamiltonian and so can be applied to many different one-dimensional models. As an example of the power of Bosonization we show how one may represent a Tomonaga-Luttinger liquid Hamiltonian of interacting, spinless Fermions as a free Boson field Hamiltonian. It is then shown how this result facilitates the calculation of various correlation functions in the XXY spin chain model.

1 Introduction

In many-particle, quantum systems, one often finds that the single particle operators used to construct a particular model are not the most convenient or the most fundamental objects to describe the system. The introduction of a new set of operators is often required to make certain complex calculations tractable. Such redefinitions are not uncommon and usually depend on the particular model under investigation.

However, when one considers only one-dimensional, Fermionic systems, there exists a universal description of the Fermion fields in terms of Boson fields that is independent of the Hamiltonian describing the system. The term *Bosonization* is used to describe such a procedure and the derivation of this fundamental relationship is the subject of the first part of this paper, § 2.

The end result is a deceptively simple equation which is easy to verify and straightforward to implement in any model. However, in order to obtain a true understanding of the nature of this relationship, we will construct the Boson representation starting from the Fermion fields themselves. Such an exercise is straightforward and transparent and gives a much clearer understanding of the mechanism at work. This first part will be written in a pedagogical style with most of the calculations performed explicitly.

In many models of interacting electrons, Bosonization greatly simplifies the calculation of certain quantities, often making otherwise difficult Fermionic derivations almost trivial in the Bosonic language. While Bosonization is used in numerous systems, we focus primarily on the Tomonaga-Luttinger liquid [1, 2] and how one may use those results to calculate certain correlation functions in the XXY spin chain model. This second part will be much less detailed than § 2 with references given to more detailed accounts.

Many useful theorems and identities are presented in Appendix A either with proof or with a reference to their proof in the literature. Other technical results which are used throughout the paper are derived in B.

Finally, we make a quick note on the mathematical conventions used throughout. We will use square brackets $[,]$ to denote a commutator of operators and curly brackets $\{,\}$ to denote an anti-commutator. Also, we use the symbol $:=$ to denote a definition of the quantity on the left-hand side and reserve the symbol $\equiv$ for a mathematical equivalence. Also, in cases where there will be no confusion, we will often omit the limits of summation. In such cases, the limits will be implied by the letter used by referring to a previous summation using that letter (*e.g.* the variable q is always summed up over all values $(2\pi/L)n_q$ with $n_q \in \mathbb{Z}^+$).

2 Bosonization formalism

While there are many reviews on Bosonization ([3, 4], for example) and even an entire book [5], our derivation will follow the careful and lucid treatment described in [6].

To prevent the reader from becoming overwhelmed by the formalism, we will first qualitatively describe the simple Bosonization procedure:

1. Starting with the two axioms listed below we define Fermion fields $\psi_\eta(x)$, $\psi_\eta^\dagger(x)$ in terms of the usual Fermion creation and annihilation

operators $c_{k\eta}^{\dagger}$, $c_{k\eta}$. These operators are then used to construct the standard multi-particle Fock space H^f of quantum states.

2. The Fermion creation and annihilation operators are then used to construct new operators $b_{q\eta}^{\dagger}$, $b_{q\eta}$ which are shown to be Bosonic. It is then proven that the Fock space H^b constructed from the b operators is the same as H^f.

3. Bosonic fields $\varphi_\eta(x)$ are then defined in terms of the b operators. It is shown that the state $\psi_\eta|\mathbf{N}\rangle_0$ (where $|\mathbf{N}\rangle_0$ is the $\mathbf{N}$ particle ground state) is an eigenvector of $b_{q\eta}$ and hence is a Bosonic coherent state. This fact is then used to derive a universal relationship between the Fermion fields ψ_η and the Boson fields φ_η.

Now that we have a good idea of the general direction we are headed, we can look at the specific details of Bosonization.

We begin with a one-dimensional Fermionic system. For reasons that will become apparent shortly, there are two requirements which must be met before we can Bosonize:

Requirement 1 *The model must be constructed in terms of Fermionic operators* $c_{k\eta}^{\dagger}$, $c_{k\eta}$ *such that*

$$\{c_{k\eta}, c_{k'\eta'}^{\dagger}\} = \delta_{kk'}\delta_{\eta\eta'}, \quad \{c_{k\eta}, c_{k'\eta'}\} = \{c_{k\eta}^{\dagger}, c_{k'\eta'}^{\dagger}\} = 0. \tag{1}$$

The k, as usual, will label momentum while $\eta \in \{1, 2, \ldots, M\}$ labels the different types of fermions in the system (*e.g.* η could label spin, right/left moving electrons, *etc.*).

Requirement 2 *The momentum eigenvalues must be discrete and unbounded.*

In general, this will be accomplished by considering a continuous line of length L and applying boundary conditions such that

$$k = \frac{2\pi}{L}(n_k - \delta_b) \quad , \; n_k \in \mathbb{Z} \tag{2}$$

where $\delta_b \in [0, 2)$ encodes the boundary conditions that are applied to fields on the line (*cf.* Eq. (5)).

For the discussion that follows we take Requirements 1 and 2 as axioms. In practice however, one usually has to do a little bit of work in order to obtain these prerequisites in a particular model. Examples of such manipulations are given in the second part of this paper, § 3.

2.1 Fermion fields and Fermionic Fock space H^f

The Fermionic operators $c^\dagger_{k\eta}$ and $c_{k\eta}$ can be Fourier transformed to define the Fermion fields

$$\psi_\eta(x) := \sqrt{\frac{2\pi}{L}} \sum_{k=-\infty}^{\infty} e^{-ikx} c_{k\eta}, \quad \psi^\dagger_\eta(x) := \sqrt{\frac{2\pi}{L}} \sum_{k=-\infty}^{\infty} e^{ikx} c^\dagger_{k\eta} \tag{3}$$

with the inverse transform given as

$$c_{k\eta} = \frac{1}{\sqrt{2\pi L}} \int_{-L/2}^{L/2} dx e^{ikx} \psi_\eta(x), \quad c^\dagger_{k\eta} = \frac{1}{\sqrt{2\pi L}} \int_{-L/2}^{L/2} dx e^{-ikx} \psi^\dagger_\eta(x). \tag{4}$$

With this definition we can now see what boundary conditions are implied by the momentum definition 2:

$$\begin{aligned}
\psi_\eta(x + L/2) &= \sqrt{\frac{2\pi}{L}} \sum_{n_k \in \mathbb{Z}} e^{-i\frac{2\pi}{L}(n_k - \frac{1}{2}\delta_b)(x+L/2)} c_{k\eta} \\
&= e^{i\frac{\pi}{2}\delta_b} \sqrt{\frac{2\pi}{L}} \sum_{n_k \in \mathbb{Z}} e^{-i\frac{2\pi}{L}(n_k - \frac{1}{2}\delta_b)x + i\pi(n_k - \frac{1}{2}\delta_b) + i\frac{\pi}{2}\delta_b} c_{k\eta} \\
&= e^{i\pi\delta_b} \sqrt{\frac{2\pi}{L}} \sum_{n_k \in \mathbb{Z}} e^{-i\frac{2\pi}{L}(n_k - \frac{1}{2}\delta_b)(x-L/2)} c_{k\eta} \\
&= e^{i\pi\delta_b} \psi_\eta(x - L/2)
\end{aligned} \tag{5}$$

where in the second equality we added and subtracted by $i\frac{\pi}{2}\delta_b$ in the exponential. Equation (5) shows us that $\delta_b = 0$ indicates periodic boundary conditions and $\delta_b = 1$ indicates anti-periodic boundary conditions.

From the definition of the ψ_η fields (3) we proceed to compute their commutators. From (1) we have trivially

$$\{\psi_\eta(x), \psi_{\eta'}(x')\} = \{\psi^\dagger_\eta(x), \psi^\dagger_{\eta'}(x')\} = 0 \tag{6}$$

and, for any $x \in (-\infty, \infty)$ we can use the definitions (3) and (2) to calculate

$$\begin{aligned}
\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} &= \frac{2\pi}{L} \sum_{k,k'=-\infty}^{\infty} e^{-i(kx - k'x')} \{c_{k\eta}, c^\dagger_{k'\eta'}\} \\
&= \delta_{\eta\eta'} \frac{2\pi}{L} \sum_{n \in \mathbb{Z}} e^{-i\frac{2\pi}{L}(x-x')(n - \frac{1}{2}\delta_b)}
\end{aligned}$$

and mapping $n \mapsto \bar{n} = -n$ and using (93) this becomes

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} = \delta_{\eta\eta'} \frac{2\pi}{L} 2\pi \sum_{\bar{n}\in\mathbb{Z}} \delta\left(\frac{2\pi}{L}(x - x' - L\bar{n})\right) e^{i(x-x')\frac{\pi}{L}\delta_b}.$$

If we use the delta function property (94) this becomes

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} = \delta_{\eta\eta'} 2\pi \sum_{\bar{n}\in\mathbb{Z}} \delta(x - x' - L\bar{n}) e^{i\pi\bar{n}\delta_b}. \tag{7}$$

For $x, x' \in (-L/2, L/2)$ this reproduces the usual Fermion field commutator (recall the 2π normalization convention) however (7) is generalized for any x, x' and manifests the appropriate boundary condition behavior.

Armed with a set of Fermion fields we now construct a Fock space on which they can act. To do this, we interpret $c^\dagger_{k\eta}$ and $c_{k\eta}$ as a creation and annihilation operators for the η-Fermions. Then define the "vacuum" state $|0, 0, \ldots, 0\rangle_0$ to be the state in which all of the $k < 0$ ($n_k < 0$) states are filled for all $\eta \in \{1, 2, \cdots, M\}$. That is

$$c_{k\eta}|\mathbf{0}\rangle_0 \quad := \quad 0, \qquad k > 0 \tag{8}$$

$$c^\dagger_{k\eta}|\mathbf{0}\rangle_0 \quad := \quad 0, \qquad k \leq 0 \tag{9}$$

where we use boldface to indicate an M dimensional array.

In order to avoid divergent expressions, we will often need to *normal order* the creation and annihilation operators such that all $c_{k\eta}$ with $k > 0$ and all $c^\dagger_{k\eta}$ with $k < 0$ appear to the right. Denoting such normal ordering by : : it is clear that

$$: A_1 A_2 \cdots A_n : = A_1 A_2 \cdots A_n - {}_0\langle\mathbf{0}|A_1 A_2 \cdots A_n|\mathbf{0}\rangle_0. \tag{10}$$

That is, the second term on the right cancels the $\delta_{kk'}$ terms that appear when you rearrange the operators. For example, if $k, k' > 0$ then, by definition, $: c_{k\eta} c^\dagger_{k'\eta'} : = -c^\dagger_{k'\eta'} c_{k\eta}$. If we apply the Fermionic commutation relations to the right hand side of (10) we get

$$\begin{aligned} c_{k\eta} c^\dagger_{k'\eta'} - {}_0\langle\mathbf{0}|c_{k\eta} c^\dagger_{k'eta'}|\mathbf{0}\rangle_0 &= \delta_{kk'}\delta_{\eta\eta'} - c^\dagger_{k'\eta'} c_{k\eta} - \delta_{kk'}\delta_{\eta\eta'} \\ &= -c^\dagger_{k'\eta'} c_{k\eta} \end{aligned}$$

which agrees with the left hand side.

The number operator for η-Fermions can now be defined as

$$\hat{N}_\eta = \sum_{k=-\infty}^{\infty} : c^\dagger_{k\eta} c_{k\eta} : \tag{11}$$

whose eigenvalues are the number of η-Fermions as measured from the Fermi level $k = -(1/2)\delta_b$ (*e.g.* $N_\eta = 2$ if all $k < 0$ η states are filled and there are two η-Fermions with $k > 0$). Let $\mathbf{N} = (N_1, N_2, \ldots, N_M) \in \mathbb{Z}^M$ describe the occupation number for all M types of Fermions. We then define the $\mathbf{N}$ particle Hilbert space $H_\mathbf{N}$ as the space spanned by all eigenvectors of $\hat{N}_\eta$ with the same value of $\mathbf{N}$, generically denoted as $|\mathbf{N}\rangle \in H_\mathbf{N}$.

We will introduce the notation

$$k_F^{N_\eta} := \frac{2\pi}{L}(N_\eta - \frac{1}{2}\delta_b) \tag{12}$$

which is the maximum momentum if N_η η-Fermions fill up the lowest k values above $k = -(1/2)\delta_b$. A state with all M Fermions filled up to $k_F^{N_\eta}$ has no particle-hole excitations and is denoted as $|\mathbf{N}\rangle_0 = |N_1, N_2, \ldots, N_M\rangle$. Hence, we can view $|\mathbf{N}\rangle_0$ as the lowest energy state in the $\mathbf{N}$-particle Hilbert space. To avoid ambiguity, we define $|\mathbf{N}\rangle_0$ explicitly as

$$|\mathbf{N}\rangle_0 := (c_1)^{N_1}(c_2)^{N_2}\cdots(c_M)^{N_M}|\mathbf{0}\rangle_0 \tag{13}$$

where

$$(c_\eta)^{N_\eta} := \begin{cases} c^\dagger_{N_\eta\eta} c^\dagger_{(N_\eta-1)\eta} \cdots c^\dagger_{1\eta} & , \; N_\eta > 0 \\ 1 & , \; N_\eta = 0 \\ c_{(N_\eta+1)\eta} c_{(N_\eta+2)\eta} \cdots c_{0\eta} & , \; N_\eta < 0 \end{cases} \tag{14}$$

where by $c_{N_\eta\eta}$ we mean $c_{k\eta}$ with $k = (2\pi/L)N_i - (1/2)\delta_b$ in accordance with the definition (2). The entire Fock space is $H = \bigoplus_\mathbf{N} H_\mathbf{N}$.

2.2 Bosonic Fock space H^b and Boson fields

Since $|\mathbf{N}\rangle_0$ is the lowest energy state in $H_\mathbf{N}$ it is clear that all other excited states in $H_\mathbf{N}$ will have particle-hole pairs. To obtain such excitations from the ground state, we must act with a $c_{k\eta}$ operator with $k < k_F^{N_\eta}$ and a $c^\dagger_{k'\eta}$

operator with $k' > k_F^{N_\eta}$. Hence, in general, we define the operators

$$b_{q\eta}^\dagger := \frac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c_{(k+q)\eta}^\dagger c_{k\eta} \tag{15}$$

$$b_{q\eta} := -\frac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c_{(k-q)\eta}^\dagger c_{k\eta} \tag{16}$$

where

$$q := \frac{2\pi}{L} n_q, \qquad n_q \in \mathbb{Z}^+. \tag{17}$$

To understand the action of these operators it is instructive to look at $b_{q\eta}^\dagger |\mathbf{N}\rangle_0$ for small values of q. Starting at $k = -\infty$ in the sum of (15), it is clear that all of the terms will give zero unless $k + q > k_F^{N_\eta}$ and $k < k_F^{N_\eta}$. For instance, if $q = 2\pi/L$ (*i.e.* $n_q = 1$) then there is only one non-zero term and we get

$$b_{(2\pi/L)\eta}^\dagger |\mathbf{N}\rangle_0 = i c_{(k_F^{N_\eta} + (2\pi/L))\eta}^\dagger c_{k_F^{N_\eta}\eta} |\mathbf{N}\rangle_0.$$

That is, the highest momentum η-Fermion is excited to the next highest momentum level. In general, $b_{q\eta}^\dagger |\mathbf{N}\rangle_0$ is a linear combination of particle-hole states, each with one of the occupied $k < k_F^{N_\eta}$ levels raised by momentum q. In this sense, we may view $b_{q\eta}^\dagger$ and $b_{q\eta}$ as creation and annihilation operators which impart or take away momentum q respectively.

With multiple uses of the commutator identities (95) and (96) we may use the definitions (15) to compute the following:

$$\begin{aligned}
[b_{q\eta}, b_{q'\eta'}] &= -\frac{1}{\sqrt{n_q n_{q'}}} \sum_{k,k'} \left([c_{(k-q)\eta}^\dagger c_{k\eta}, c_{(k'-q')\eta'}^\dagger] c_{k'\eta'} + c_{(k'-q')\eta'}^\dagger [c_{(k-q)\eta}^\dagger c_{k\eta}, c_{k'\eta'}] \right) \\
&= -\frac{1}{\sqrt{n_q n_{q'}}} \sum_{k,k'} \left(c_{(k-q)\eta}^\dagger \{c_{k\eta}, c_{(k'-q')\eta'}^\dagger\} c_{k'\eta'} - c_{(k'-q')\eta'}^\dagger \{c_{(k-q)\eta}^\dagger, c_{k'\eta'}\} c_{k\eta} \right) \\
&= -\frac{1}{\sqrt{n_q n_{q'}}} \sum_{k,k'} \left(c_{(k-q'-q)\eta}^\dagger c_{k'\eta} \delta_{\eta\eta'} - c_{(k'-q'-q)\eta}^\dagger c_{k\eta} \delta_{\eta\eta'} \right) \\
&= 0.
\end{aligned} \tag{18}$$

A similar calculation also yields

$$[b_{q\eta}^\dagger, b_{q'\eta'}^\dagger] = 0 \tag{19}$$

and

$$[b_{q\eta}, b^\dagger_{q'\eta'}] = \frac{1}{\sqrt{n_q n'_q}}\delta_{\eta\eta'}\sum_k \left(c^\dagger_{(k+q'-q)\eta}c_{k\eta} - c^\dagger_{(k+q')\eta}c_{(k+q)\eta}\right). \tag{20}$$

We must be careful with this last equality as it involves the subtraction of two different infinite sums (whereas before, in (18), we basically had an infinite sum of zeros). For $q \neq q'$ the $c^\dagger$ and c operators have different momentum values making the summation finite so that we can shift the summation in the second term $k \mapsto k - q$ which gives

$$[b_{q\eta}, b^\dagger_{q'\eta'}] = 0, \qquad q' \neq q.$$

For $q = q'$ the $c^\dagger$ and c operators have the same momentum rendering the summation infinite so that we cannot be sure that the cancellation will work.[1] To clarify the situation, we employ the normal ordering expression (10) to get

$$\begin{aligned}[b_{q\eta}, b^\dagger_{q'\eta'}] &= \frac{1}{n_q}\delta_{\eta\eta'}\delta_{qq'}\sum_k \Big(: c^\dagger_{k\eta}c_{k\eta} : - : c^\dagger_{(k+q)\eta}c_{(k+q)\eta} : \\ &\quad + {}_0\langle \mathbf{0}|c^\dagger_{k\eta}c_{k\eta}|\mathbf{0}\rangle_0 - {}_0\langle \mathbf{0}|c^\dagger_{(k+q)\eta}c_{(k+q)\eta}|\mathbf{0}\rangle_0\Big).\end{aligned}$$

We can now shift the index of the second term $k \mapsto k - q$ with impunity since the normal ordering will rearrange the terms appropriately depending on whether $k + q \gtrless k_F^{N_\eta}$ and we can be assured that the result will be the same as the first term. The last two terms simply count particles in the vacuum state so we get

$$[b_{q\eta}, b^\dagger_{q'\eta'}] = \frac{1}{n_q}\delta_{\eta\eta'}\delta_{qq'}\left(\sum_{k=-\infty}^{0} - \sum_{k=-\infty}^{-n_q}\right)$$

where the two sums simply add up[2] to n_q so that we finally get

$$[b_{q\eta}, b^\dagger_{q'\eta'}] = \delta_{qq'}\delta_{\eta\eta'}. \tag{21}$$

[1]For $q = q'$ one may be tempted to perform the same shift in the second term of (20) and cancel the two terms to get zero. However, since the action of $c_{k,\eta}$ on $|\mathbf{N}\rangle_0$ (for instance) depends on whether k is greater than or less than $k_F^{N_\eta}$, such a shift will not necessarily produce the same term as the first.

[2]Notice how this result depends crucially on Requirement 2.

Hence we have shown that $b^\dagger_{q\eta}$ and $b_{q\eta}$ are Bosonic operators.

From the definitions (15) it is clear that

$$b_{q\eta}|\mathbf{N}\rangle_0 = 0 \tag{22}$$

since, for a given k in the infinite sum, $c_{k\eta}$ acting on $|\mathbf{N}\rangle_0$ only gives a non-zero contribution when $k < k_F^{N_\eta}$ in which case it annihilates the k momentum level. However, the $c^\dagger_{(k-q),\eta}$ operator attempts to create a particle at the $k-q$ level which is already filled since $q > 0$. In this sense, we view $|\mathbf{N}\rangle_0$ as a Bosonic vacuum state. Normal ordering of Boson operators can then be defined as shifting all b operators to the right of $b^\dagger$ operators. Clearly the relation (10) still holds for Bosonic normal ordering.

Given a Bosonic vacuum state $|\mathbf{N}\rangle_0$, we can construct a Bosonic Fock space of states $H^b_{\mathbf{N}}$ in the usual way by repeated application of the creation operator $b^\dagger_{q\eta}$. The direct sum of all such $\mathbf{N}$-particle spaces is denoted as H^b. The remarkable fact is that this is the same space of states as H^f which is the subject of the following

Theorem 1 *The Bosonic Fock space H^b is identical to the Fermionic Fock space H^f in that any vector $|\mathbf{N}\rangle \in H^f_{\mathbf{N}}$ is in H^b and vice versa.*

Sketch of Proof: (full proof given in [6]) The proof that $H^f = H^b$ follows from showing that $H^f_{\mathbf{N}} = H^b_{\mathbf{N}}$ for all $\mathbf{N}$. Since the b operators (15) and (16 are defined as functions of the creation and annihilation operators of $H^f_{\mathbf{N}}$ it is clear that $H^b_{\mathbf{N}} \subset H^f_{\mathbf{N}}$. Hence, it is sufficient to show that the "number" of states in $H^b_{\mathbf{N}}$ is the same as the "number" of states in $H^f_{\mathbf{N}}$.

To show that the two spaces are the same "size" one can compute the grand-canonical partition function for some Hamiltonian theory using both $H^b_{\mathbf{N}}$ and $H^f_{\mathbf{N}}$. Since the partition function is defined as the sum of positive, definite quantities, and since $H^b_{\mathbf{N}} \subset H^f_{\mathbf{N}}$ the two calculations will only yield the same result if $H^b_{\mathbf{N}} = H^f_{\mathbf{N}}$. Otherwise, the larger space would yield a larger result. This partition function calculation has been done in Appendix B of [6] and indeed yields the same result for both $H^b_{\mathbf{N}}$ and $H^f_{\mathbf{N}}$.

$\blacksquare$

Given this result, we will simply denote the Fock space as $H = \bigoplus_{\mathbf{N}} H_{\mathbf{N}}$ since there is no difference between the one created by Boson operators and the one created by Fermionic operators. The usefulness of this theorem comes from the fact that any vector $|\mathbf{N}\rangle \in H_{\mathbf{N}}$ will be of the form $f(\{b^\dagger\})|\mathbf{N}\rangle_0$ for some function f of the $b^\dagger$ operators.

If we restrict our attention only to the Bosonic creation and annihilation operators, the $H_{\mathbf{N}}$ spaces are disconnect spaces since no combination of b operators will take a state from $H_{\mathbf{N}}$ to $H_{\mathbf{N}'}$ for $\mathbf{N} \neq \mathbf{N}'$. To facilitate such mappings we define ladder operators (herein called *Klein factors* in conjunction with the literature) F_η, $F_\eta^\dagger$ which lower and raise N_η by one respectively.

These Klein factors are unambiguously defined by the following properties

$$[F_\eta, b_{q\eta'}] = [F_\eta, b^\dagger_{q\eta'}] = [F^\dagger_\eta, b_{q\eta'}] = [F^\dagger_\eta, b^\dagger_{q\eta'}] = 0 \tag{23}$$

(*i.e.* all F operators commute with all b operators) and, if $|\mathbf{N}\rangle := f(\{b^\dagger\})|\mathbf{N}\rangle_0$, then

$$\begin{aligned} F^\dagger_\eta|\mathbf{N}\rangle &:= f(\{b^\dagger\})c^\dagger_{(N_\eta+1)\eta}|N_1,\ldots,N_\eta,\ldots,N_M\rangle \\ &= f(\{b^\dagger\})T_\eta|N_1,\ldots,N_\eta+1,\ldots,N_M\rangle \end{aligned} \tag{24}$$

$$\begin{aligned} F_\eta|\mathbf{N}\rangle &:= f(\{b^\dagger\})c_{N_\eta\eta}|N_1,\ldots,N_\eta,\ldots,N_M\rangle \\ &= f(\{b^\dagger\})T_\eta|N_1,\ldots,N_\eta-1,\ldots,N_M\rangle \end{aligned} \tag{25}$$

where

$$T_\eta := (-1)^{\sum_{\bar\eta=1}^{\eta-1}\hat{N}_{\bar\eta}}$$

gives a plus or minus sign depending on how many Fermionic operators c and $c^\dagger$ must commute past before reaching the N_η term in the definition of $|\mathbf{N}\rangle_0$ (*cf.* Eq. (13)).

From this definition, it should be clear that

$$F^\dagger_\eta F_\eta = F_\eta F^\dagger_\eta = 1. \tag{26}$$

This relation is quite intuitive: the result of lowering and then raising (or raising then lowering) a given N_η leaves you with the same state.

The final ingredient required to prove the Bosonization identity is the definition of Boson fields $\varphi^\dagger_\eta(x)$ and $\varphi_\eta(x)$:

$$\varphi_\eta(x) := -\sum_{q>0}\frac{1}{\sqrt{n_q}}e^{-iqx}b_{q\eta}e^{-\frac{a}{2}q} \tag{27}$$

$$\varphi^\dagger_\eta(x) := -\sum_{q>0}\frac{1}{\sqrt{n_q}}e^{iqx}b^\dagger_{q\eta}e^{-\frac{a}{2}q} \tag{28}$$

so that the Hermitian combination is

$$\begin{aligned}\phi_\eta(x) &:= \varphi_\eta(x) + \varphi_\eta^\dagger(x) \\ &= -\sum_{q>0} \frac{1}{\sqrt{n_q}} \left(e^{-iqx} b_{q\eta} + e^{iqx} b_{q\eta}^\dagger\right) e^{-\frac{a}{2}q}. \end{aligned} \tag{29}$$

The a is a regularization factor in order to prevent ultra-violet divergences as $q \to \infty$. In some sense it can be thought of as the lattice spacing but for our purposes it is simply a mathematical regulator. Since $q = (2\pi/L)n_q$ we see that ϕ is periodic in $x \in (-L/2, L/2)$.

In the following section we will show that there is a fundamental relationship between the fields $\psi_\eta(x)$ and the fields $\phi_\eta(x)$. Whenever this relationship is used (such as in the examples in the second half of this paper) one will need to know a number of properties of the $\varphi_\eta(x)$ fields, such as their various commutators. Some of these properties are derived in Appendix B. While these properties are not needed for the proof of the Bosonization identity in the next section, they will be required when we look at applications in § 3.

2.3 The Bosonization identity

After all of this work in defining a multitude of operators and vector spaces the proof of the Bosonization identity is fairly straightforward.

We begin by using the definitions (3) and (15) and the commutation property (96) to compute

$$\begin{aligned}[b_{q\eta'}, \psi_\eta(x)] &= -\frac{i}{\sqrt{n_q}} \sqrt{\frac{2\pi}{L}} \sum_{k,k'} e^{-ik'x} [c^\dagger_{(k-q)\eta'} c_{k\eta'}, c_{k'\eta}] \\ &= \frac{i}{\sqrt{n_q}} \sqrt{\frac{2\pi}{L}} \sum_{k,k'} e^{-ik'x} \{c^\dagger_{(k-q)\eta'}, c_{k'\eta}\} c_{k\eta'} \\ &= \delta_{\eta\eta'} \frac{i}{\sqrt{n_q}} e^{iqx} \sqrt{\frac{2\pi}{L}} \sum_{k,k'} e^{-ikx} c_{k\eta} \\ &= \delta_{\eta\eta'} \alpha_q(x) \psi_\eta(x) \end{aligned} \tag{30}$$

where

$$\alpha_q(x) := \frac{i}{\sqrt{n_q}} e^{iqx}. \tag{31}$$

Acting with both sides of (30) on the vacuum state $|\mathbf{N}\rangle_0$ and using the fact that $b_{q\eta}$ annihilates the vacuum (22) we get

$$b_{q\eta'}\psi_\eta(x)|\mathbf{N}\rangle_0 = \delta_{\eta\eta'}\alpha_q(x)\psi_\eta(x)|\mathbf{N}\rangle_0. \tag{32}$$

We see that $\psi_\eta(x)|\mathbf{N}\rangle_0$ is an eigenvector of the Boson annihilation operator $b_{q\eta}$ with an eigenvalue $\delta_{\eta\eta'}\alpha_q(x)$. Hence, $\psi_\eta(x)|\mathbf{N}\rangle_0$ can be represented as a Boson coherent state in the ground state $F_\eta\lambda_\eta(x)|\mathbf{N}\rangle_0$

$$\psi_\eta(x)|\mathbf{N}\rangle_0 = \exp\left(\sum_{q>0}\alpha_q(x)b^\dagger_{q\eta}\right)F_\eta\lambda_\eta(x)|\mathbf{N}\rangle_0 \tag{33}$$

where $\lambda_\eta(x)$ is a phase factor to be determined. The reason why we must use $F_\eta\lambda_\eta(x)|\mathbf{N}\rangle_0$ and not just simply $|\mathbf{N}\rangle_0$ is due to the fact that $\psi_\eta(x)$ lowers the N_η number by one which, as discussed previously, cannot be accomplished by any combination of b operators. The fact that we are not using $|\mathbf{N}\rangle_0$ requires the inclusion of the λ_η phase factor. Comparing the expression (33) with the definition (28) (in the limit $a \to 0$) yields

$$\psi_\eta(x)|\mathbf{N}\rangle_0 = e^{-i\varphi^\dagger(x)}F_\eta\lambda_\eta(x)|\mathbf{N}\rangle_0. \tag{34}$$

The calculation of $\lambda_\eta(x)$ is a simple exercise in applying the properties of the various operators. We accomplish this by calculating the quantity ${}_0\langle\mathbf{N}|F^\dagger_\eta\psi_\eta(x)|\mathbf{N}\rangle_0$ in two ways. First, we use the fact that $F^\dagger_\eta$ commutes with all b operators so that

$${}_0\langle\mathbf{N}|F^\dagger_\eta\psi_\eta(x)|\mathbf{N}\rangle_0 = {}_0\langle\mathbf{N}|\exp\left(\sum_{q>0}\alpha_q(x)b^\dagger_{q\eta}\right)F^\dagger_\eta F_\eta\lambda_\eta(x)|\mathbf{N}\rangle_0.$$

However, using (26) and the fact that $b_{q\eta}|\mathbf{N}\rangle_0 = 0 \Rightarrow {}_0\langle\mathbf{N}|b^\dagger_{q\eta} = 0$ we obtain

$$\lambda_\eta(x) = {}_0\langle\mathbf{N}|F^\dagger_\eta\psi_\eta(x)|\mathbf{N}\rangle_0. \tag{35}$$

If we sub the Fourier expansion of $\psi_\eta(x)$ (3) into the right hand side of (35) we get

$$\lambda_\eta(x) = \sqrt{\frac{2\pi}{L}}\,{}_0\langle\mathbf{N}|F^\dagger_\eta\sum_k e^{-ikx}c_{k\eta}|\mathbf{N}\rangle_0.$$

We now note that ${}_0\langle\mathbf{N}|F_\eta^\dagger$ doesn't contain any particle-hole pairs. Hence, the only non-zero term in the summation comes from $c_{k_F^{N_\eta}\eta}|\mathbf{N}\rangle_0 = F_\eta|\mathbf{N}\rangle_0$, *i.e.* the annihilation of the highest η momentum level. The result is

$$\lambda_\eta(x) = \sqrt{\frac{2\pi}{L}} e^{-ik_F^{N_\eta}x}\ {}_0\langle\mathbf{N}|F_\eta^\dagger F_\eta|\mathbf{N}\rangle_0$$

and using (26) once again yields the desired result

$$\lambda_\eta(x) = \sqrt{\frac{2\pi}{L}} e^{-ik_F^{N_\eta}x}. \tag{36}$$

The Bosonization identity is nearly at hand! We simply must generalize (34) for general states $\psi_\eta(x)|\mathbf{N}\rangle$ where $|\mathbf{N}\rangle = f(\{b^\dagger\})|\mathbf{N}\rangle_0$ is a general vector in $H_\mathbf{N}$. To accomplish this, we will need the following two identities:

$$\psi_\eta(x) f(\{b_{q\eta'}^\dagger\}) = f(\{b_{q\eta'}^\dagger - \delta_{\eta\eta'}\alpha_q^*(x)\})\psi_\eta(x) \tag{37}$$

$$e^{-i\varphi_\eta(x)} f(\{b_{q\eta'}^\dagger\}) e^{i\varphi_\eta(x)} = f(\{b_{q\eta}^\dagger - \delta_{\eta\eta'}\alpha_q^*(x)\}). \tag{38}$$

The first follows from (30) and applying (99) with $A = b_{q\eta'}^\dagger - \delta_{\eta\eta'}\alpha_q^*(x)$, $B = \psi_\eta(x)$, and $D = \delta_{\eta\eta'}\alpha_q^*(x)$. The second is a direct application of (30) and (98) with $B = \varphi_\eta(x)$ and $A = b_{q\eta'}^\dagger$.

And now for the home stretch. Let $|\mathbf{N}\rangle = f(\{b_{q\eta}^\dagger\})|\mathbf{N}\rangle_0$ so that we can compute $\psi_\eta(x)|\mathbf{N}\rangle$. Using (37) to compute the ψ past f we get

$$\psi_\eta(x)|\mathbf{N}\rangle = f(\{b_{q\eta'}^\dagger - \delta_{\eta\eta'}\alpha_q^*(x)\})\psi_\eta(x)|\mathbf{N}\rangle_0$$

and subbing in the Boson coherent state (34) gives

$$\psi_\eta(x)|\mathbf{N}\rangle = f(\{b_{q\eta'}^\dagger - \delta_{\eta\eta'}\alpha_q^*(x)\})e^{-i\varphi_\eta^\dagger(x)} F_\eta \lambda_\eta(x)|\mathbf{N}\rangle_0.$$

We can then commute the $e^{-i\varphi_\eta^\dagger(x)}$ and F_η operator all the way to the left (by (19) and (23)) and using (38) yields

$$\psi_\eta(x)|\mathbf{N}\rangle = F_\eta\lambda_\eta(x)e^{-i\varphi_\eta^\dagger(x)}e^{-i\varphi_\eta(x)} f(\{b_{q\eta'}^\dagger\})e^{i\varphi_\eta(x)}|\mathbf{N}\rangle_0.$$

Looking at the far-right exponential we note that, since b annihilates the vacuum, only the zeroth order term contributes. Hence, we can re-institute the definition of $|\mathbf{N}\rangle$ to get

$$\psi_\eta(x)|\mathbf{N}\rangle = F_\eta\lambda_\eta(x)e^{-i\varphi_\eta^\dagger(x)}e^{-i\varphi_\eta(x)}|\mathbf{N}\rangle.$$

Since this is true for all states $|\mathbf{N}\rangle$ we arrive (finally!) at the Bosonization identity

$$\psi_\eta(x) = F_\eta \lambda_\eta(x) e^{-i\varphi_\eta^\dagger(x)} e^{-i\varphi_\eta(x)}. \tag{39}$$

Having produced our desired result, let us pause to discuss some of the properties and implications of (39).

- Although the final Bosonization identity appears simple, it is not at all obvious *a priori*. For example, if we look at the Boson coherent state (34), the left hand side is a simple, linear combination of states, each with a *single* particle-hole pair. At first site, the right hand side, when written explicitly as a function of the c and $c^\dagger$ operators, seems like it would be a complicated combination of all sorts of states with any number of particle-hole pairs. The fact that only the single hole states contribute to this mess is indeed miraculous.

- The Bosonization identity, as derived above, is a very general result, true for all values of x and L. Furthermore, since we did not require the use of a specific Hamiltonian, eq. (39) is a *model independent* operator identity, true for any one dimensional system subject to Requirements 1 and 2. Once the identity has been implemented, however, one can take the $L \to \infty$ as we will see in § 3.

- The Bosonization identity (39) is normal ordered and therefore independent of the value of the regularization parameter a. However, one can cast the identity in an even simpler form by making use of the a-dependent product (102) so that

  $$\psi_\eta(x) = \frac{1}{\sqrt{a}} F_\eta e^{-i\Phi_\eta(x)} \tag{40}$$

  where

  $$\Phi_\eta(x) := \phi_\eta(x) + \frac{2\pi}{L}\left(\hat{N}_\eta - \frac{1}{2}\delta_b\right) x \tag{41}$$

  which simply reduces to $\phi_\eta(x)$ in the $L \to \infty$ limit (recall the definition of ϕ from (29)).

3 Applications

Now that we have fully derived the Bosonization identity (39) we will now proceed to describe it's application to a few specific models so that the reader may understand it's usefulness. Contrary to the previous exposition, the following will not contain detailed calculations but will simply sketch the various derivations giving reference to other sources when more information is required.

3.1 Free electrons with linear dispersion

Our exposition follows that given in § 7 of [6]. We consider a collection of N_η Fermions for each type $\eta \in \{1, 2, \ldots, M\}$ freely propagating in a one dimensional system of length L with normalized linear dispersion relation

$$\epsilon(k) = k. \tag{42}$$

The boundary conditions are determined by the value of $\delta_b \in [0, 2)$ as discussed in § 2.1. Clearly such a system satisfies Requirements 1 and 2.

As we will see later on, such dispersions are considered when one wishes to approximate the behavior of the particles about the Fermi energy. Although such a system is quite simple to solve and does not, necessarily, require the power of Bosonization, we state the results of Bosonization here so that we may refer to them later on.

The Hamiltonian for this system is

$$H^0 = \sum_{\eta=1}^{M} H^0_\eta$$

where H^0_η is the free η-particle Hamiltonian

$$H^0_\eta = \sum_{k=-\infty}^{\infty} k : c^\dagger_{k\eta} c_{k\eta} : \tag{43}$$

where the c operators are the Fermionic creation and annihilation operators as above. In terms of the Fermion fields (3) one can show that

$$H^0_\eta = \int_{-L/2}^{L/2} \frac{dx}{2\pi} : \psi^\dagger_\eta(x) i\partial_x \psi_\eta(x) : . \tag{44}$$

Our goal now is to express H^0_η in terms of the Boson field $\phi_\eta(x)$. To this end we will derive two properties of H^0_η which will fully determine the Bosonic form of H^0_η.

From (43) we see that the η-Hamiltonian is proportional to the number operator and hence particle number is conserved. This means that the $\mathbf{N}$ particle ground state $|\mathbf{N}\rangle_0$ is an eigenvector of H^0_η. After a short calculation one can show that this ground state energy is

$$E^0_\eta(\mathbf{N}) := {}_0\langle\mathbf{N}|H^0_\eta|\mathbf{N}\rangle_0 = \frac{\pi}{L}N_\eta(N_\eta + 1 - \delta_b). \tag{45}$$

This is the first required property.

The second comes from using the definition (15) to compute the commutator

$$[H^0_\eta, b_{q\eta'}] = q\delta_{\eta\eta'}b^\dagger_{q\eta}. \tag{46}$$

Equation (45) together with (46) show that the state $b^\dagger_{q\eta}|\mathbf{N}\rangle_0$ has energy $E = E^0_\eta(\mathbf{N}) + q$ indicating that $b^\dagger_{q\eta}|\mathbf{N}\rangle_0$ represents a state with momentum q greater than the ground state as discussed in the previous section.

It is then quite clear that equations (45) and (46) dictate the Bosonic form of H^0_η to be

$$\begin{aligned} H^0_\eta &= \sum_{q>0} q b^\dagger_{q\eta} b_{q\eta} + \frac{\pi}{L}\hat{N}_\eta(\hat{N}_\eta + 1 - \delta_b) & (47) \\ &= \int_{-L/2}^{L/2} \frac{dx}{2\pi}\frac{1}{2} : (\partial_x\phi_\eta(x))^2 : + \frac{\pi}{L}\hat{N}_\eta(\hat{N}_\eta + 1 - \delta_b). & (48) \end{aligned}$$

It is a straightforward task to substitute the definition of $\phi_\eta(x)$ (29) into (48) and confirm this result. We will utilize this result in what follows.

3.2 Tomonaga-Luttinger liquids

Tomonaga-Luttinger liquid theory is a model for interacting electrons in a one-dimensional system. To Bosonize such a model, our first task will be to manipulate the model so as to conform to Requirements 1 and 2. We will do so by first consider only the free electron model before introducing an interaction.

We characterize the electrons by their momentum p and consider the free particle dispersion relation

$$\epsilon(p) = \frac{p^2 - p_F^2}{2m} \tag{49}$$

where p_F is the Fermi momentum and m the mass of an electron. Clearly such a dispersion is not suitable for Bosonization since a state such as $|\mathbf{0}\rangle_0$ would have infinite, positive energy since all states with $p < 0$ would be filled.

To remedy this situation we divide the electrons into two types: left moving, labelled by L, and right moving, labelled by R. If we define c_p and $c_p^\dagger$ as the electron annihilation and creation operators then we can subdivide the Fermion field as follows

$$\begin{aligned}\Psi(x) &:= \sqrt{\frac{2\pi}{L}}\sum_{p=-\infty}^{\infty} e^{ipx}c_p \\ &= \sqrt{\frac{2\pi}{L}}\sum_{k=-k_F}^{\infty}(e^{-i(k+k_F)}c_{kL} + e^{i(k+k_F)x}c_{kR} \qquad (50) \\ & \qquad (51)\end{aligned}$$

where we have defined the creation and annihilation operators for left and right moving electrons as follows

$$c_{kL} := c_{-(k+k_F)}, \qquad c_{kR} := c_{k+k_F}. \qquad (52)$$

We will use ν to label either L or R and $\epsilon_{k,\nu}$ the dispersion relation in terms of k.

Since $k \in (-k_F, \infty)$ and so bounded from below, our system does not meet Requirement 2. We can remedy this situation by introducing an infinite set of negative energy states $k < -k_F$ with linear dispersion. In this way we have $k \in (-\infty, \infty)$ with dispersion

$$\epsilon_{k,\nu} = \begin{cases} \epsilon(k+k_F) & , \ k > -k_F \\ \epsilon(0) + v_F(k+k_F) & , \ k < -k_F \end{cases} \qquad (53)$$

where v_F is some constant. The inclusion of such negative energy states will not effect the physics of low-energy excitations about k_F so long as the perturbations (due either to electron interactions or external fields) remain small compared to k_F. In this way, the perturbations will never be strong enough to excite a negative energy state and so they remain "invisible" as far as our formalism is concerned.

We can then use the left and right annihilation operators to define left and right electron fields, *a la* equation (3)

$$\psi_{L/R}(x) := \sqrt{\frac{2\pi}{L}}\sum_{k=-\infty}^{\infty} e^{\mp ikx}c_{k(L/R)}. \qquad (54)$$

Comparing this definition with (50) we see that the original Fermion field can be written as

$$\Psi(x) \approx e^{-ik_F x}\psi_L(x) + e^{ik_F x}\psi_R(x) \tag{55}$$

where the approximate sign is used since we are now including the negative energy momentum states on the right hand side. For definiteness, we assume anti-periodic boundary conditions in the Fermion operators. As shown above, this means we take $\delta_b = 1$.

The definition of the Boson fields and other quantities in § 2.2 and Appendix B follow exactly as stated except for right moving quantities where we must include an additional minus sign due to the minus sign in the exponential in the definition (54) of $\psi_R(x)$. We may still use the derived results so long as we make the following substitutions for all right moving quantities:

$$\begin{aligned} x &\mapsto -x \\ \partial_x &\mapsto -\partial_x . \end{aligned}$$

Quoting equations (29), (40), (103), and (104) we get

$$\phi_{L/R}(x) = -\sum_{q>0} \frac{1}{\sqrt{n_q}} \left(e^{\mp iqx} b_{q(L/R)} + e^{\pm iqx} b^\dagger_{q(L/R)} \right) e^{-\frac{a}{2}q} \tag{56}$$

$$\psi_{L/R}(x) = \frac{1}{\sqrt{a}} F_{L/R} e^{-i\phi_{L/R}(x) \mp i\frac{2\pi}{L}(\hat{N}_{L/R}+\frac{1}{2})} \tag{57}$$

$$\rho_{L/R}(x) = \frac{1}{2\pi} : \psi^\dagger_{L/R}(x)\psi_{L/R}(x) : = \pm\frac{1}{2\pi}\partial_x\phi_{L/R}(x) + \frac{1}{L}\hat{N}_{L/R} \tag{58}$$

where, as before, $q = (2\pi/L)n_q$ with $n_q \in \mathbb{Z}^+$.

The Hamiltonian for this free theory is that of (44) in terms of the Fermion fields

$$H^0_\eta = \int_{-L/2}^{L/2} \frac{dx}{2\pi} : \left(\psi^\dagger_L(x) i\partial_x \psi_L(x) - \psi^\dagger_R(x) i\partial_x \psi_R(x) \right) : \tag{59}$$

and (48) in terms of the Boson fields

$$H^0 = \sum_{\nu=L,R} \left[\int_{-L/2}^{L/2} \frac{dx}{2\pi} \frac{1}{2} : (\partial_x \phi_\nu(x))^2 : + \frac{\pi}{L} \hat{N}^2_\nu \right] . \tag{60}$$

Comparison with (58) indicates that we can write the latter Hamiltonian as

$$H^0 = \pi \int_{-L/2}^{L/2} dx \ : \left(\rho^2_L(x) + \rho^2_R(x) \right) : . \tag{61}$$

We now consider a local electron-electron interaction of the form

$$H^I := 2\pi \int_{-L/2}^{L/2} dx : \left(g_2 \rho_L(x) \rho_R(x) + \frac{1}{2} g_4 (\rho_L^2(x) + \rho_R^2(x)) \right) : \qquad (62)$$

and, after a bit of algebra, one can show that the total Hamiltonian can be written as

$$\begin{aligned} H &= H^0 + H^I \\ &= v \frac{\pi}{2} \int_{-L/2}^{L/2} dx : \left(\frac{1}{g} (\rho_L(x) + \rho_R(x))^2 + g (\rho_L(x) - \rho_R(x))^2 \right) : \qquad (63) \end{aligned}$$

where

$$v := \sqrt{(1+g_4)^2 - g_2^2} \qquad (64)$$

$$g := \sqrt{\frac{1+g_4-g_2}{1+g_4+g_2}}. \qquad (65)$$

After all of this work of casting the Hamiltonian into a Bosonic form we are now able to see the great benefit of Bosonization. Looking at the form of the Hamiltonian (63) and comparing it with the two definitions of the density (58) we see that the Hamiltonian contains terms with products of four Fermion ψ fields but is *quadratic* in the Boson fields ϕ. This is a beautiful result since it is known that such quadratic Hamiltonians are easily diagonalized by a Bogolyubov transformation of the Bosonic creation and annihilation operators $b^\dagger$ and b.

This has been done in detail in [6] and so we simply quote the results. The transformation is of the form

$$B_{q\pm} = \frac{1}{\sqrt{8}} \left\{ \left(\frac{1}{\sqrt{g}} + \sqrt{g} \right) (b_{qL} \mp b_{qR}) \pm \left(\frac{1}{\sqrt{g}} - \sqrt{g} \right) (b_{qL}^\dagger \mp b_{qR}^\dagger) \right\} \qquad (66)$$

with associated number operators

$$\hat{N}_\pm = \frac{1}{2} (\hat{N}_L \mp \hat{N}_R). \qquad (67)$$

The Boson fields associated to the new Boson creation and annihilation operators (66) are[3]

$$\begin{aligned}\Phi_\pm(x) &= \frac{1}{\sqrt{8}}\left\{\left(\frac{1}{\sqrt{g}}+\sqrt{g}\right)(\phi_L(x)\mp\phi_R(-x))\right.\\ &\quad\left.\pm\left(\frac{1}{\sqrt{g}}-\sqrt{g}\right)(\phi_L(-x)\mp\phi_R(x))\right\}. \end{aligned} \tag{68}$$

With our Hamiltonian diagonal our calculation is complete. Hence, we may safely take the $L \to \infty$ limit and enjoy the wonderfully simple result

$$H = \frac{v}{2}\sum_{\nu=\pm}\int_{-\infty}^{\infty}\frac{dx}{2\pi} :(\partial_x\Phi_\nu(x))^2: . \tag{69}$$

By using the technique of Bosonization, we have been able to represent the interacting Hamiltonian (63) by a free Bosonic field Hamiltonian (69). All of the information about the Fermionic system is stored in the scaling factor v which encodes the relative strengths of the interactions and serves to re-scale the boson fields. To calculate any Fermionic quantity (such as a correlation function) we have simply to use the Bosonization identity to cast the quantity in terms of Bosonic fields and use the known results from free Boson theory, replacing the fields ϕ with the re-scaled fields $\phi/\sqrt{v}$. The following section gives an explicit example of such a calculation.

3.3 XXY spin chain

The outline of this section is as follows: we will present the XXY spin chain model in the usual spin operator basis and show how one may transform it into a Luttinger liquid form (63) and thus easily facilitating the calculation of various spin-spin correlation functions. This exposition partially follows both [7] and [8].

Our system is that of a one-dimensional lattice with a single electron at each lattice site with only spin degrees of freedom. The Hamiltonian for the

[3]Some care should be taken to note that both Φ operators are manifestly left moving in that, in the Heisenberg picture, these fields depend only on the combination $x+t$ which is different from ϕ_R, for instance, which would depend on $x-t$. More details may be found in [6]. We mention this only in passing as such concerns are not important for our purposes.

XXY spin chain is

$$H = -\frac{1}{2} J \sum_{i=1}^{N} \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right) + J_z \sum_{i=1}^{N} S_i^z S_{i+1}^z \tag{70}$$

where S_i^a, $a = x, y, z$ are the spin operators for the lattice site i and $S_i^\pm := S_i^x \pm i S_i^y$ are the raising and lowering operators for site i which raise and lower the spin in the z direction by 1. We note that the sign of J is arbitrary since a unitary spin rotation of π changes the sign of the first coefficient. For convenience we take $J > 0$. On the other hand, the sign of J_z is important as it governs whether adjacent spins tend to align or anti-align depending on which state lowers the energy ($J_z > 0$ tends to anti-alignment while $J_z < 0$ tends to alignment).

We can now perform a transformation to new variables (the so-called Jordan-Wigner transformation) f_i, $f_i^\dagger$ defined as

$$\begin{aligned} S_i^+ &= f_i^\dagger K(i) = K(i) f_i^\dagger && (71) \\ S_i^- &= K(i) f_i = f_i K(i) && (72) \\ S_i^z &= f_i^\dagger f_i - \frac{1}{2} && (73) \end{aligned}$$

where

$$\begin{aligned} K(i) &:= \exp\left[i\pi \sum_{j=1}^{i-1} f_j^\dagger f_j \right] && (74) \\ &= \exp\left[i\pi \sum_{j=1}^{i-1} \left(S_j^z + \frac{1}{2} \right) \right]. && (75) \end{aligned}$$

From this last equality (the first part of which is just a spin rotation by π about the z axis) and the fact that a spin rotation by 2π gives -1 we see that $K^2 = 1$ and hence, from (71) and (72) we get

$$\begin{aligned} f_i^\dagger &= S_i^+ K(i) && (76) \\ f_i &= S_i^- K(i). && (77) \end{aligned}$$

From these relations and the usual SU(2) spin algebra one can compute [7]

$$\{f_i, f_j^\dagger\} = \delta_{ij}, \qquad \{f_i, f_j\} = \{f_i^\dagger, f_j^\dagger\} = 0 \tag{78}$$

indicating that the f fields are Fermionic.

The transformed Hamiltonian then has the form

$$H = -\frac{J}{2}\sum_i \left(f_i^\dagger f_{i+1} + f_{i+1}^\dagger f_i\right) + J_z \sum_i \left(f_i^\dagger f_i - \frac{1}{2}\right)\left(f_{i+1}^\dagger f_{i+1} - \frac{1}{2}\right) \quad (79)$$

For $J_z = 0$ the system is invariant under translations and we can Fourier transform our fields to get

$$f_k = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{-ink} f_n \quad (80)$$

$$f_k^\dagger = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{ink} f_n^\dagger \quad (81)$$

so that the $J_z = 0$ Hamiltonian has the form of a free electron theory [7]

$$H^0 = \sum_k \epsilon(k) f_k^\dagger f_k \quad (82)$$

with

$$\epsilon k = -J\cos k \quad (83)$$

(note that distances are measured in units of the lattice spacing a).

To determine the $J_z = 0$ ground state (*i.e.* the state which gives zero total magnetization) we can sum both sides of (73) over all N lattice sites to get

$$S^z_{\text{tot}} = M - \frac{N}{2} \quad (84)$$

where M is the total number of electrons since $f_i^\dagger f_i$ is the number operator. Hence we see that the ground state is at half filling $M = N/2$ so that, from the dispersion relation (83), we determine that the ground state Fermi energy is $E_F = 0$ with Fermi momentum $k_F = \pm\pi/2$.

In order to Bosonize this Fermion theory we will need to transform this model into one on a continuous space. To this end we will consider low energy excitations about E_F and linearize the dispersion relation about k_F

$$\epsilon(k) \approx \pm(k - k_F). \quad (85)$$

In the regime close to k_F we can define the continuous analogue $\psi(x)$ of f_i to be

$$\Psi(x) \approx e^{i\frac{\pi}{2}x}\psi_R(x) + e^{-i\frac{\pi}{2}x}\psi_L(x). \quad (86)$$

That is, only the two $k_F = \pm\pi/2$ Fourier modes are important for low energy excitations. Comparing this with (55) we see that ψ_R and ψ_L are of the same form as the right and left moving electron fields of § 3.2.

Hence we may use our previous result (61) and write

$$H^0 = J \int dx : \left(\rho_L^2(x) + \rho_R^2(x)\right) : \tag{87}$$

where we now use $\rho_\nu(x) =: \psi_\nu^\dagger(x)\psi_\nu(x) :$ since our fields are no longer defined with the 2π normalization.

In order to express the J_z term of (79) we substitute $\Psi(x)$ defined in (86) for f_n and using the normal order expression (10) we can write

$$\begin{aligned} f_n^\dagger f_n - \frac{1}{2} &= : f_n^\dagger f_n : \\ &= : \psi_R^\dagger \psi_R : + : \psi_L^\dagger \psi_L : + (-1)^n \left(: \psi_R^\dagger \psi_L : + : \psi_L^\dagger \psi_R : \right) \end{aligned} \tag{88}$$

and using the definition for the density ρ_ν we can write the J_z term in the Hamiltonian (79) as

$$\begin{aligned} H^z &= J_z \int dx \left[(\rho_L + \rho_R)^2 - (\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)^2 \right] \\ &= J_z \int dx \left[(\rho_L^2 + \rho_R^2) + 4\rho_L \rho_R - 2\left((\psi_L^\dagger \psi_R)^2 + (\psi_R^\dagger \psi_L)^2 \right) \right]. \end{aligned} \tag{89}$$

For the moment let us neglect the last ψ terms in the previous expression and come back to them later. In this case, if we divide through by J we see that the H^z Hamiltonian has the same form as the Tomonaga-Luttinger liquid interaction Hamiltonian (62) with $g_2 \propto J_z/J$ and $g_4 \propto J_z/J$. Since the free Hamiltonian H^0 is of the same form as (61) we can write the Hamiltonian in the form (63) with [8]

$$v = 1 + 2\frac{J_z}{J}. \tag{90}$$

Hence we can use the same Bogolyubov transformation as in § 3.2 and write H as a free, scaled Boson scalar Hamiltonian (69).

The result is that we can use the known correlation functions for free Boson theory to calculate correlation functions for the XXY spin chain model under consideration. All we have to do is rescale the fields

$$\Phi \mapsto \frac{1}{R}\Phi$$

where

$$R := \sqrt{v} = \sqrt{1 + 2\frac{J_z}{J}}.$$

This has been done in [8] for the spin-z correlation function and the spin-$\pm$ correlation function with the results

$$\begin{aligned} < S^z(x,t) S^z(0,0) > &= -\frac{1}{16\pi^3 R^2}\left[\frac{1}{(x-t)^2} + \frac{1}{(x+t)^2}\right] \\ &\quad + \text{const.}(x^2 - t^2)^{-\frac{1}{4\pi R^2}} \end{aligned} \tag{91}$$

and

$$\begin{aligned} < S^+(x,t) S^-(0,0) > &\propto (-(x+t)(x-t))^{-\pi R^2} \\ &\quad + (-1)^x \text{const.}(-(x-t)(x+t))^{-\frac{1}{4\pi}(\frac{1}{R} - 2\pi R)^2} \\ &\quad \times \left(\frac{1}{(x-t)^2} + \frac{1}{(x+t)^2}\right). \end{aligned} \tag{92}$$

It turns out that these results are good only for $J_z \ll J$ which is when we can safely neglect the last terms in (89). It turns out that one can solve the problem exactly by employing the Bethe ansatz [7, 8]. The final result is simply a different expression for the scaling parameter R given now by

$$R = \sqrt{2 - \frac{2}{\pi}\cos^{-1}\frac{J_z}{J}}$$

so that the correlation functions (91) and (92) still hold.

4 Summary

In this paper we have rigourously proven a general representation of Fermion fields in terms of Boson fields in one-dimensional systems. This relationship depends only on the fact that the momentum be discrete and unbounded which is often easy to accomplish in any given model. Hence, such Bosonization procedures provide an additional tool in calculating a number of quantities in one-dimensional Fermionic systems, often making the derivation tractable if not trivial.

As an example of the power of Bosonization we were able to exactly calculate the spin-z and spin-$\pm$ (or transverse) correlation functions in the XXY spin chain model. This was done by first transforming the Hamiltonian to a Tomonaga-Luttinger liquid form which was already diagonalized using the Bosonization procedure. In effect, we were able to map the interacting Hamiltonian to a free Boson theory for which much is known. All of the physics of the Fermionic interaction was encoded in the rescaling of the Boson field. Hence, the seemingly intractable problem of calculating the exact correlation function in terms of Fermion fields is made tractable by the implementation of Bosonization.

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A Useful theorems and identities

Here we collect a variety of theorems and identities used throughout the paper. Many of these results are well-known and so no attempt has been made to achieve any sort of rigour.

Delta Function Properties
In order to extend the validity of fields beyond $x \in (-L/2, L/2)$ it is necessary to use the generalized Fourier representation of the delta function [9]

$$\sum_{n\in\mathbb{Z}} e^{inx} = 2\pi \sum_{\bar{n}\in\mathbb{Z}} \delta(x - 2\pi\bar{n}). \tag{93}$$

This gives you one δ function in each region $x \in (2\pi\bar{n}, 4\pi\bar{n})$.

Another useful property of the delta function is

$$\delta(kx) = \frac{1}{|k|}\delta(x) \tag{94}$$

where k is a constant real number. To see this, let $f(x)$ be any function and compute

$$\begin{aligned} \int_{-\infty}^{\infty} dx f(x)\delta(kx) &= \pm \int_{-\infty}^{\infty} \frac{dy}{k} f(y/k)\delta(y) \\ &= \frac{1}{|k|} f(0) \end{aligned}$$

where the upper sign corresponds to $k > 0$ and the lower sign to $k < 0$. Here we see that the form given on the right side of (94) gives the same result.

Operator Identites
For operators A, B and C the following (anti-)commutator identities can easily be proven by expanding out both sides of the equality

$$[A, BC] = [A, B]C + B[A, C] \tag{95}$$

$$[A, BC] = \{A, B\}C - B\{A, C\}. \tag{96}$$

The proofs of the following theorems are given in Appendix C of [6]. Here, A, B, C, and D are elements of an operator algebra on an arbitrary vector space.

Theorem 2 *If* $[A, [A, B]] = [B, [A, B]] = 0$ *then*

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{97}$$

and

$$e^{-B} f(A) e^B = f(A + [A, B]) \tag{98}$$

where f is any function of the operators.

Theorem 3 *If* $[A, B] = DB$ *and* $[A, D] = [B, D] = 0$ *then*

$$f(A)B = Bf(A + D). \tag{99}$$

B Some properties of the Boson Fields

Here we derive some properties of the Boson operators $\varphi_\eta(x)$ and $\varphi_\eta^\dagger(x)$ that were used in the text.

First we will calculate the commutators between the φ fields. Using the definitions (27) and (28) and the Boson commutation relationship between the Boson creation and annihilation operators (21) and (18) it is immediately clear that

$$[\varphi_\eta(x), \varphi_{\eta'}(x')] = [\varphi_\eta^\dagger(x), \varphi_{\eta'}^\dagger(x')] = 0. \tag{100}$$

The commutator between φ_η and $\varphi_\eta^\dagger$ is a little bit trickier. Utilizing the definitions again we get

$$\begin{aligned}
[\varphi_\eta(x), \varphi_{\eta'}^\dagger(x')] &= \sum_{q,q'} \frac{1}{\sqrt{n_q n_q'}} e^{-i(qx - q'x') - \frac{a}{2}(q+q')} [b_{q\eta}, b_{q'\eta'}^\dagger] \\
&= \delta_{\eta,\eta'} \sum_{n_q=1}^{\infty} \frac{1}{n_q} e^{-\frac{2\pi}{L} n_q (i(x-x')+a)}
\end{aligned}$$

where we have used the definition $q = (2\pi/L)n_q$. Defining

$$\alpha := -i\frac{2\pi}{L}(x - x' - ia)$$

we can write the commutator as

$$\begin{aligned}
[\varphi_\eta(x), \varphi_{\eta'}^\dagger(x')] &= \delta_{\eta\eta'} \sum_{n=1}^{\infty} \frac{1}{n} e^{\alpha(x,x')n} \\
&= \delta_{\eta\eta'} \int d\alpha \sum_{n=1}^{\infty} e^{\alpha n} \\
&= \delta_{\eta\eta'} \int d\alpha \frac{1}{1 - e^{\alpha}} \\
&= -\delta_{\eta\eta'} \ln\left(1 - e^{-i\frac{2\pi}{L}(x - x' - ia)}\right).
\end{aligned}$$

In the limit $L \to \infty$ we can expand the exponential to lowest order in $1/L$ to get

$$[\varphi_\eta(x), \varphi_{\eta'}^\dagger(x')] \approx -\delta_{\eta\eta'} \ln\left(i\frac{2\pi}{L}(x - x' - ia)\right). \tag{101}$$

This commutator is very useful when wishing to calculate such quantities as

$$\begin{aligned}
e^{i\phi_\eta^\dagger(x)} e^{i\phi_\eta(x)} &= e^{i(\varphi_\eta^\dagger(x) + \varphi_\eta(x))} e^{\frac{1}{2}[i\varphi_\eta^\dagger(x), i\varphi_\eta(x)]} \\
&= \sqrt{\frac{L}{2\pi a}} e^{i\phi_\eta(x)}
\end{aligned} \tag{102}$$

where, in the first equality, we made use of (97) and in the second we used the Hermitian combination (29) of the boson fields.

A quantity that we will find very useful is the density operator

$$\rho_\eta(x) := \frac{1}{2\pi} : \psi^\dagger_\eta(x)\psi_\eta(x) : \tag{103}$$

where the 2π comes from the 2π normalization of the Fermion fields. To express this in terms of boson fields we substitute the definition of the fermion fields

$$\begin{aligned}\rho_\eta(x) &= \frac{1}{L}\sum_{k,k'} e^{-i(k'-k)x} : c^\dagger_{k\eta}c_{k'\eta} : \\ &= \frac{1}{L}\sum_{q=-\infty}^{\infty} e^{-iqx} \sum_{k=-\infty}^{\infty} : c^\dagger_{(k-q)\eta}c_{k\eta} :\end{aligned}$$

where in the last equality we simply shifted the summation $k \mapsto k' - q$.

We can break the q summation into three terms with $q < 0$, $q = 0$, and $q > 0$ to get

$$\begin{aligned}\rho_\eta(x) &= \frac{1}{L}\Bigg\{\sum_{q>0} e^{+iqx} \sum_{k=-\infty}^{\infty} : c^\dagger_{(k+q)\eta}c_{k\eta} : + \sum_{q>0} e^{-iqx} \sum_{k=-\infty}^{\infty} : c^\dagger_{(k-q)\eta}c_{k\eta} : \\ &\quad + \sum_{k=-\infty}^{\infty} : c^\dagger_{k\eta}c_{k\eta} : \Bigg\}\end{aligned}$$

where we simply mapped the $q < 0$ summation using $q \mapsto -q$. Using the definition of the Boson raising and lowering operators (15) and (16) and the number operator we can write this as

$$\rho_\eta(x) = \frac{1}{L}\sum_{q>0}\left(-i\sqrt{n_q}b^\dagger_{q\eta} + i\sqrt{n_q}b_{q\eta}\right) + \frac{1}{L}\hat{N}_\eta.$$

Comparing this expression with the definition of the $\phi_\eta(x)$ (29) and it's derivative we arrive at our desired expression

$$\rho_\eta(x) = \frac{1}{2\pi}\partial_x\phi_\eta(x) + \frac{1}{L}\hat{N}_\eta. \tag{104}$$

In the limit $L \to \infty$ only the first term survives. The fact that ρ_η is quadratic in Fermion fields ψ but linear in the Boson field ϕ is the key to the usefulness of Bosonization.

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