The Mapping from 2D Ising Model to Quantum Spin Chain

Mandy Man Chu Wong*
Department of Physics & Astronomy,
University of British Columbia,
Vancouver, B.C. Canada, V6T 1Z1
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I. INTRODUCTION

This paper discusses the method of studying classical d+1 dimensional Ising Model as a d dimension quantum spin system and vice-versa [5]. It can be shown, in the Transfer Matrix Formalism, that the two models are equivalent in the time continuum limit of the classical Ising model. I will first discuss this mapping in the simplest context - the one dimensional Ising model. Then we will look at the two dimensional Ising Model and show that it is equivalent to an one dimensional quantum spin chain. I will also comment on the general correspondences of the critical phenomena under this mapping. The reader can find detailed discussion of this topic in [5] and [6].

II. THE TRANSFER MATRIX FORMALISM FOR A SINGLE QUANTUM SPIN

This section basically derives the Feynman-Kac formula presented earlier in this course [3] by looking at the one dimensional Ising chain. Recall that for time-independent systems, the Feynman-Kac formula links between statistical mechanics and the quantum evolution of the system. The partition function is given by:

$$\mathcal{Z} = Tr[e^{-\beta \hat{H}}] = \sum dx \sum_n e^{-\beta E_n} \langle x|n\rangle\langle n|x\rangle = \sum dx \mathcal{G}(x, -i\hbar\beta; x, 0) \tag{1}$$

Consider an one dimensional Ising chain with N sites and an Ising variable $\sigma_n^z = \pm 1$ on each site n with periodic boundary condition. This is a purely statistical system with 2^N configurations $\{\sigma_n^z\}$. The partition function is given by:

$$\mathcal{Z} = \sum_{\{\sigma_n^z\}} e^{-\beta E(\{\sigma_n^z\})} \tag{2}$$

$$E(\{\sigma_n^z\}) = \sum_{n=1}^{N} (-J\sigma_n^z \sigma_{n+1}^z - h\sigma_n^z) \tag{3}$$

where $\sum_{\{\sigma_n^z\}}$ means $\prod_{n=1}^N \sum_{\sigma_n^z = \pm 1}$. The parameter h represents an external magnetic field. For convenience, I will absorb the temperature into the couplings J and h. One should keep in mind that $J, h \propto \frac{1}{T_{IM}}$. This partition function can be evaluated exactly following the original solution of Ising [1]. The trick is to write $\mathcal{Z}$ as a trace over a matrix product, with one matrix for every site on the chain.

$$E(\{\sigma_n^z\}) = \sum_n \mathcal{L}(n, n+1) \tag{4}$$

$$\mathcal{L}(n, n+1) = \frac{J}{2}[(\sigma_n^z - \sigma_{n+1}^z)^2 - h(\sigma_n^z + \sigma_{n+1}^z)] \tag{5}$$

$$\mathcal{Z} = \sum_{\{\sigma_n^z\}} e^{-\sum_n \mathcal{L}(n,n+1)} = \sum_{\{\sigma_n^z\}} \prod_n e^{-\mathcal{L}(n,n+1)} = \sum_{\{\sigma_n^z\}} \prod_n T(n, n+1) \tag{6}$$

* mandyman@phas.ubc.ca

I should note that the energy expression in (5) is different from (3) by a constant but this is not important when we calculate the partition function. The next step is to interpret $T(n,n+1) = e^{-\mathcal{L}(n,n+1)}$ as a matrix element of the two different states located at site n and n+1. $T(n,n+1) \equiv <\sigma^z_n|\mathbf{T}|\sigma^z_{n+1}>$. $\mathbf{T}$ is named the **Transfer Matrix**. In this problem, each Ising variable has two states ($\sigma^z_n = \pm 1$), so $\mathbf{T}$ is only a 2X2 matrix and is very easy to solve.

$$T(\uparrow,\uparrow) = e^{h} \;\; T(\uparrow,\downarrow) = e^{-2J} \;\; T(\downarrow,\uparrow) = e^{-2J} \;\; T(\downarrow,\uparrow) = e^{-h} \tag{7}$$

$$\mathbf{T} = \begin{bmatrix} e^{h} & e^{-2J} \\ e^{-2J} & e^{-h} \end{bmatrix} \tag{8}$$

The partition function can now takes the following form.

$$\begin{aligned} \mathcal{Z} &= \sum_{\sigma^z_1=\pm1} \cdots \sum_{\sigma^z_N=\pm1} \langle\sigma^z_1|\mathbf{T}|\sigma^z_2\rangle\langle\sigma^z_2|\mathbf{T}|\sigma^z_3\rangle \cdots \langle\sigma^z_N|\mathbf{T}|\sigma^z_{N+1}\rangle & (9)\\ &= \sum_{\sigma^z_1}\langle\sigma^z_1|\mathbf{T}(\sum_{\sigma^z_2}|\sigma^z_2\rangle\langle\sigma^z_2|)\mathbf{T}(\sum_{\sigma^z_3}|\sigma^z_3\rangle\langle\sigma^z_3|)\cdots(\sum_{\sigma^z_N}|\sigma_N\rangle\langle\sigma_N|)\mathbf{T}|\sigma^z_{N+1}\rangle & (10)\\ &= \sum_{\sigma^z_1}\langle\sigma^z_1|\mathbf{T}^N|\sigma^z_{N+1}\rangle = (periodic)\sum_{\sigma^z_1}\langle\sigma^z_1|\mathbf{T}^N|\sigma^z_1\rangle = Tr[\mathbf{T}^N] & (11) \end{aligned}$$

If we imagine that the axis of the lattice is the time axis of quantum mechanics, then $\mathbf{T}$ carries information from one time to a neighboring time. We can identify the transfer matrix as the time evolution operator for a quantum system of a single spin. Recall that for time-independent Hamiltonian, the propagator has the form $\mathcal{G}(\sigma^z_f, t_f, \sigma^z_i, t_i) = \langle\sigma^z_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|\sigma^z_0\rangle$. We can associate the transfer matrix with the propagator in the following way:

$$\mathbf{T}^N \leftrightarrow e^{\frac{-i(t_f-t_i)}{\hbar}\hat{H}} = e^{-\frac{\epsilon}{\hbar}\hat{H}N} \tag{12}$$

where $\epsilon = i\frac{t_f-t_i}{\hbar}$ is the imaginary time step and $\hat{H}$ is the Hamiltonian of the single spin quantum system. Using the fact that $e^{\epsilon(O_1+O_2)} = e^{\epsilon O_1}e^{\epsilon O_2}(1+\mathcal{O}(\epsilon^2))$, we can write Eqn. (12) in the limit $\epsilon \to 0$ as

$$\mathbf{T} = e^{-\frac{\epsilon}{\hbar}H} \approx 1 - \frac{\epsilon}{\hbar}H \tag{13}$$

This is not true in general for the Ising system, but there exists a limit for the parameters J and h such that it is consistent with the requirement $\epsilon \to 0$ in the quantum system. If we choose $e^{-2J} = \epsilon$ and $h = \lambda e^{-2J} = \lambda\tau$ where λ is a fixed constant, then the transfer matrix becomes

$$\mathbf{T} = \begin{bmatrix} 1+\lambda\epsilon & \epsilon \\ \epsilon & 1-\lambda\epsilon \end{bmatrix} = 1 + \epsilon(\hat{\sigma}^x + \lambda\hat{\sigma}^z) \tag{14}$$

where $\hat{\sigma}^x$ and $\hat{\sigma}^z$ are the Pauli matrices. The limit $\epsilon \to 0$ corresponds to when $J, h \to 0$ which is the low temperature limit of the Ising system as $J, h \propto \frac{1}{T_{IM}}$. The quantum Hamiltonian is

$$\hat{H} = \hat{\sigma^x} + \lambda\hat{\sigma^z} \tag{15}$$

Now that we have established the connection between the two systems, we can look at the correlation function of the Ising system and its correspondences to the quantum system.

III. THE CORRESPONDENCES BETWEEN CLASSICAL STATISTICAL MECHANICS AND THE EQUIVALENT EUCLIDEAN QUANTUM SYSTEM

In this section, I will use the classical Ising chain to show the general correspondences between the classical and quantum system under this mapping. Although, this simple model does not have any phase transitions, it is still worth examining as there are regions in which the correlation "length" ξ becomes very large; the properties of these regions are very similar to those in the vicinity of the phase transition points in higher dimensions. For simplicity, I will only consider the case when h=0. For periodic boundary condition ($\sigma_1^z = \sigma_{N+1}^z$), the partition function become the trace of the transfer matrix.

$$\mathcal{Z} = \sum_{\sigma_1^z} \langle \sigma_1^z | \mathbf{T}^N | \sigma_{N+1}^z \rangle = Tr[\mathbf{T}^N] \tag{16}$$

$$= \lambda_1^N + \lambda_2^N \tag{17}$$

where $\lambda_1 = 2e^{-J}\cosh(J)$ and $\lambda_2 = 2e^{-J}\sinh(J)$ is the eigenvalue of the transfer matrix. The *two-point spin correlator* is defined to be

$$\langle \sigma_i^z \sigma_j^z \rangle = \frac{1}{\mathcal{Z}} \sum_{\{\sigma_n^z\}} e^{-E(\{\sigma_n^z\})} \sigma_i^z \sigma_j^z \tag{18}$$

$$= \frac{1}{\mathcal{Z}} Tr[\mathbf{T}^i \hat{\sigma}^z \mathbf{T}^{j-i} \hat{\sigma^z} \mathbf{T}^{N-j}] \tag{19}$$

The second line in the above equation assumes $j \geq i$ and $\hat{\sigma^z}$ is the Pauli matrix. The trace can be evaluated in closed form in the basis in which $\mathbf{T}$ is diagonal. The eigenvectors of $\mathbf{T}$ are $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ which is exactly that of the $\hat{\sigma}^x$ Pauli matrix. Using the matrix elements $\langle \uparrow_x | \sigma^z | \uparrow_x \rangle = \langle \downarrow_x | \sigma^z | \downarrow_x \rangle = 0$ and $\langle \uparrow_x | \sigma^z | \uparrow_x \rangle = \langle \downarrow_x | \sigma^x | \uparrow_x \rangle = 1$, we get

$$\langle \sigma_i^z \sigma_j^z \rangle = \frac{\lambda_1^{N-j+i} \lambda_2^{j-i} + \lambda_2^{N-j+i} \lambda_1^{j-i}}{\lambda_1^N + \lambda_2^N} \tag{20}$$

In the infinity chain limit ($N \to \infty$) , the correlation becomes $\langle \sigma_i^z \sigma_j^z \rangle = \tanh(J)^{j-i}$. Again if we interpret the lattice spacing as the imaginary time step $\tau_n = n\epsilon$, then the correlation becomes

$$\langle \sigma^z(\tau_j) \sigma^z(\tau_i) \rangle = e^{-\frac{|\tau_j - \tau_i|}{\xi}} \tag{21}$$

where ξ is the correlation length and is given by $\frac{1}{\xi} = \frac{1}{a} \ln \cosh(K)$. There are two remarks I would like to make:

(1) The quantum Hamiltonian at zero external field can be written as $H = \frac{\Delta}{2} \hat{\sigma}^x$ where $\Delta = 2\frac{e^{-2J}}{\epsilon}$. Then in the limit when the classical and the quantum model coincide ($J \to \infty$), the gap is inversely proportional to the correlation length.

$$\Delta = \frac{1}{\xi} \tag{22}$$

As mentioned before, there is no phase transition in this lowest dimensional case, but from the above relation, the critical behavior of the classical model corresponds exactly to that of the quantum model. When the correlation length become infinite ($\xi \to \infty$), Δ becomes gapless. Although I have showed eqn (22) using the simplest model, this result is general and can be applied to all other system using the same mapping. One of the main interests will be the character of the lattice theory's phase diagrams and the nature of their critical regions.

(2) My second remark is the correspondence in the correlation function. The *time-ordered* correlator, G, of the quantum system in imaginary time is

$$G(\tau_1, \tau_2) = \begin{cases} \frac{1}{\mathcal{Z}} Tr[e^H \hat{\sigma}^z(\tau_1) \hat{\sigma}^z(\tau_2)] & \tau_1 > \tau_2 \\ \frac{1}{\mathcal{Z}} Tr[e^H \hat{\sigma}^z(\tau_2) \hat{\sigma}^z(\tau_1)] & \tau_1 < \tau_2 \end{cases} \tag{23}$$

where $\hat{\sigma}^z(\tau)$ is defined to be the imaginary time evolution. $\hat{\sigma}^z(\tau) \equiv e^{\hat{H}\tau} \hat{\sigma}^z e^{-\hat{H}\tau}$ Upon carrying the mapping described above, one should find that

$$G(\tau_1, \tau_2) = \lim_{\epsilon \to 0} \langle \sigma^z(\tau_1) \sigma^z(\tau_2) \rangle \tag{24}$$

The correspondences between classical statistical mechanics and the equivalent Euclidean quantum system are summarized in Table (1).

Quantum System	Statistical System
Ground state	Equilibrium state
Propagator	Correlation function
Mass gap	Reciprocal of the correlation length

TABLE I: General correspondences between statistical and quantum system using the τ-continuum approach

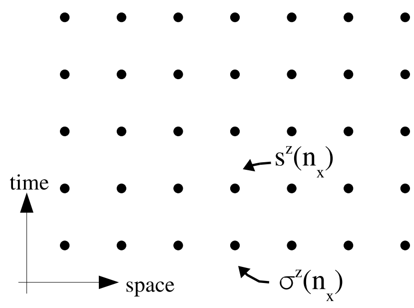

FIG. 1: The space-time lattice of the two dimensional Ising model. $s^z(n)$ and $\sigma^z(n)$ are spin variables on adjacent spatial rows. The argument n runs over the integers labeling the sites of a spatial row.

IV. THE TWO-DIMENSIONAL ISING MODEL AND THE QUANTUM SPIN CHAIN

We are now ready to study the two dimensional case and illustrate the general remarks of the previous section. Consider a 2D classical Ising model with Ising variables $\sigma^z(\vec{n}) = \pm 1$ on each site $\vec{n} = (n_x, n_t)$. Denote the unit lattice vector in the temporal direction by $\hat{\tau}$ and that in spatial direction by $\hat{x}$ as shown in Fig. (1) The Action or the energy function of this statistical system is:

$$S = -\sum_{\vec{n}}[\beta_\tau \sigma^z(\vec{n}+\hat{\tau})\sigma^z(\vec{n}) + \beta\sigma^z(\vec{n}+\hat{x})\sigma^z(\vec{n})] \tag{25}$$

The system is anisotropic with different temporal and spatial couplings (β_τ and β). We will follow the same procedure by first constructing the transfer matrix and then find the τ-continuum Hamiltonian of this model. It is better to write the Action in a symmetric form.

$$S = \frac{1}{2}\beta_\tau \sum_{\vec{n}}(\sigma^z(\vec{n}+\hat{\tau}) - \sigma^z(\vec{n}))^2 - \beta\sum_{\vec{n}}\sigma^z(\vec{n}+\hat{x})\sigma^z(\vec{n})] \tag{26}$$

Consider two neighboring spatial rows and label the spin variables in one row $\sigma^z(n_x)(\equiv \sigma^z(n_x, n_t))$ and those in the next row $s^z(n_x)(\equiv \sigma^z(n_x, n_t+1))$. The argument n_x runs from 1 to M_x, where M_x is the number of site on each spatial row. The action can now be written as a sum over these rows

$$S = \sum_{n_t} L(n_t+1, n_t) \tag{27}$$

$$L(n_t+1, n_t) = \frac{1}{2}\beta_\tau \sum_{n_x}(s^z(n_x) - \sigma^z(n_x))^2 - \frac{1}{2}\beta\sum_{n_x}(\sigma^z(n_x+1)\sigma^z(n_x) + \sigma^z(n_x+1)s^z(n_x)) \tag{28}$$

where I have suppressed the n_t index. Again, one could define $T(n_t+1, n_t) \equiv e^{-L(n_t+1,n_t)}$ and identify it as a matrix element $\langle\{s^z(n_x)\}|\mathbf{T}|\{\sigma^z(n_x)\}\rangle$. There are 2^{M_x} possible spin configurations on each row, so the transfer matrix will be a $2^{M_x}\text{X}2^{M_x}$ matrix. We do not need to solve for $\mathbf{T}$ explicitly, instead one could look at the individual matrix elements to come up with the scaling limit for $\beta\tau m\beta$ and the quantum Hamiltonian. For the diagonal element of the matrix, $s^z(n_x) = \sigma^z(n_x)$ for all n_x, and L becomes

$$L(0\ flips) = -\beta\sum_{n_x}\sigma^z(n_x+1)\sigma^z(n_x) \tag{29}$$

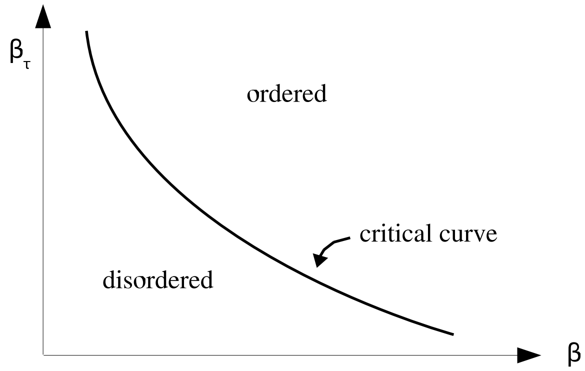

FIG. 2: The phase diagram of the classical two-dimensional Ising Model.

If there is one spin flipped between the two rows,then

$$L(1\ flip) = 2\beta_\tau - \frac{1}{2}\beta \sum_{n_x} [\sigma^z(n_x+1)\sigma^z(n_x) + s^z(n_x+1)s^z(n_x)] \tag{30}$$

and if there were n spin flips,

$$L(n\ flips) = 2n\beta_\tau - \frac{1}{2}\beta \sum_{n_x} [\sigma^z(n_x+1)\sigma^z(n_x) + s^z(n_x+1)s^z(n_x)] \tag{31}$$

Next is to determine how the couplings β_τ and β should be scaled so that the transfer matrix has the form $\mathbf{T} = e^{\tau\hat{H}} \approx 1 - \tau\hat{H}$ as $\tau \to \infty$. Consider the various matrix elements of $\mathbf{T}$

$$\mathbf{T}(0\ flips) = e^{\beta\sum_{n_x}\sigma^z(n_x+1)\sigma^z(n_x)} \approx 1 - \tau\hat{H}|_{0\ flips} \tag{32}$$

$$\mathbf{T}(1\ flip\ ) = e^{2\beta_\tau} e^{\frac{1}{2}\beta\sum_{n_x}[\sigma^z(n_x+1)\sigma^z(n_x)+s^z(n_x+1)s^z(n_x)]} \approx -\tau\hat{H}|_{1\ flip} \tag{33}$$

$$\mathbf{T}(n\ flips) = e^{2n\beta_\tau} e^{\frac{1}{2}\beta\sum_{n_x}[\sigma^z(n_x+1)\sigma^z(n_x)+s^z(n_x+1)s^z(n_x)]} \approx -\tau\hat{H}|_{n\ flips} \tag{34}$$

One can identity $\tau = e^{-2\beta_\tau}$ and $\beta = \lambda e^{2\beta_\tau} = \lambda\tau$ to obtain a condition that is consistent with $\tau \to 0$. Therefore, as $\tau \to 0$, $\beta_\tau \to \infty$. This tells us that in order for the physics to be the same in this lattice formulation,the couplings must be adjusted appropriately. We see that taking a τ-continuum limit forces us to consider very anisotropic statistical systems. The quantum Hamiltonian can be interpreted as an Ising model in a transverse magnetic field.

$$\hat{H} = -\sum_m \hat{\sigma}^x(m) - \lambda \sum_m \sigma^z(m+1)\sigma^z(m) \tag{35}$$

In the next section, I will discuss the physical meaning of λ the phase diagram and critical region of the "classical" two-dimensional Ising model.

V. CRITICAL REGION OF THE CLASSICAL TWO-DIMENSIONAL ISING MODEL

The two-dimensional Ising model has a phase transition. In the space of the parameters β_τ, β there is a critical curve which separates the ordered (ferromagnetic) and disordered (paramagnetic) phases. This is shown in Fig. (2). The critical curve is given by [2]

$$\sinh(2\beta_\tau)\sinh(2\beta) = 1 \tag{36}$$

The form of the critical curve in the limit as $\beta_\tau \to \infty$ is $\beta = e^{-2\beta_\tau}$ which has the same scaling relation found earlier except that λ has the specific value of 1. This shows that we can view the τ-continuum version of the theory as a natural limiting case of the general model, and that the parameter λ can be used to label its phase. One can then relate the temperature of the Ising model to a unique quantum model with a corresponding value of λ. If $\lambda > 1 (< 1)$, the Ising model lies in the disordered(ordered) phase. So one can associate $T_{IM} \propto \frac{1}{\lambda}$, where the critical temperature corresponds to when $\lambda = 1$.

VI. CONCLUSION

I have shown how zero and one dimensional quantum Ising model is equivalent to one and two dimensional statistical Ising model with one dimension being identified as the evolution in time. These mapping can be generalized to other systems with more than one component to the Ising variables. The general relations between the classical and the quantum system discussed in section three remain unchanged. In particular, the relation that the mass gap $\propto \frac{1}{\xi}$ allow one to identify the critical behavior from one system to another. However, when studying the classical system from the quantum side, one can only obtain results for highly anisotropic lattice system.

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[1] E. Ising,*Z. Phys.* **31**, 253 (1925)
[2] T. Schultz, D. Matis, and E. Lieb, *Rev. Mod. Phys.* **36**, 856 (1964)
[3] M. Berciu, Lecture notes for Phys503, UBC *http://www.phas.ubc.ca/ berciu/* (2005)
[4] S. Sachdev, *Quantum Phase Transition* (Cambridge University Press, NY) (1999)
[5] J. Kogut *Rev. Mod. Phys.* **51**, 659 (1979)
[6] E. Fradkin, L. Susskind, *Phys. Rev. D* **17**, 2637 (1979)