Conductance from Transmission Probability

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(Dated: November 25, 2005)

I. Introduction

For large conductors, conductance follows Ohm's Law G= σW/L where G = conductance, σ = conductivity, which is independent of the sample dimensions, W = width of the conductor, and L = length of the conductor. Therefore, this law predicts that when L goes to 0, G goes to infinity. Unfortunately, G^{-1} experimentally approaches a minimum value of G_C^{-1} which will be explained to be contact resistance. A new conductance formula: G = $(2e^2/h)$MT where M is the number of modes, and T is the probability that an electron transmits through the conductor, is shown to produce the correct conductance for conductors of both large and small scales. In this paper, the new conductance formula will be derived along explaining its ramifications to electron energy distribution, voltage drop, and heat dissipation across conductors.

II. Ballistic Conductor

A. Contact Resistance

To discover what is contact resistance, we will examine the setup shown in figure 1. There are two contacts joined by a conductor of length L and width W. At the contacts, there is an infinite number of transverse modes that carry the current, while in the conductor, there are only a few modes. The redistribution of current at the interface is what leads to a minimum resistance called contact resistance G_C^{-1}.

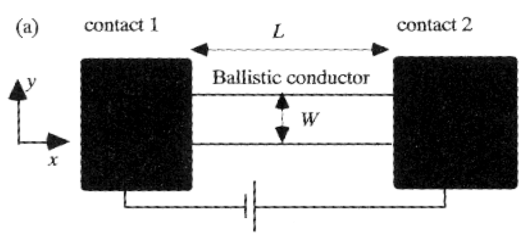

Figure 1: Setup where conductor is placed between two contacts

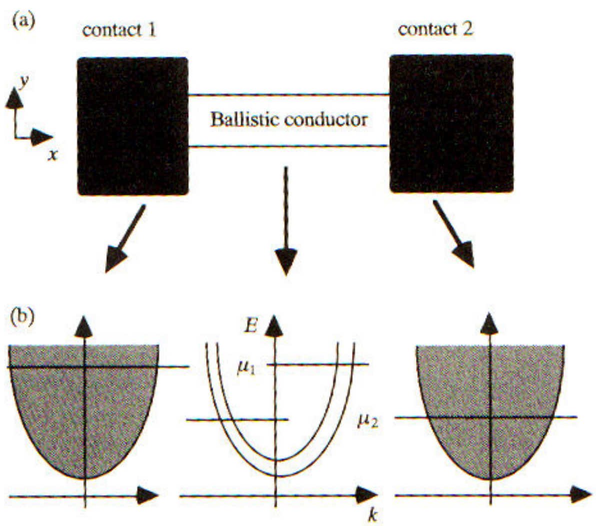

Figure 2: There is an infinite number of modes in the contacts, but only a few modes in the conductor

B. Derivation of Contact Resistance

We will now derive the value of G_C making two assumptions: ballistic conductor and reflectionless contacts. For a ballistic conductor, the probability of transmission for electrons through the conductor is unity. If the contacts are reflectionless, there are no reflections of the electrons entering the contacts from the conductor, but there is still reflection of the electrons entering the conductor from the contacts. Reflectionless contacts are shown by numerical calculations to be a good approximation as long as the electrons are not too close to the bottom of the energy band.

We now look at current when we have apply a bias across the contacts, μ_1 on the left contact, contact 1, and μ_2 on the right contact, contact 2. The +k states in the conductor will have energy μ_1 and the -k states in the conductor will have energy μ_2. The reason for this is if we apply the same energy μ_1 on both contacts, obviously the +k and –k states will have the same potential. Now, let us change the potential on contact to μ_2. The –k states from the second contact will have the new potential, but the energy of the +k states will not be affected since there is no causal relationship between the +k and –k states due to reflectionless contacts. As well, the current is carried only by the occupied states between μ_1 and μ_2 since the +k and –k states below the lower energy will cancel.

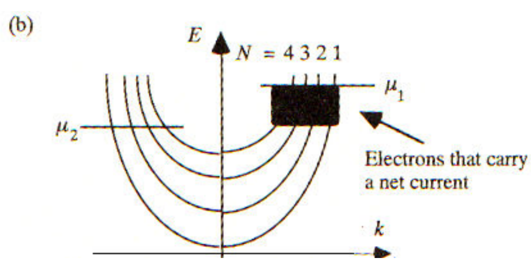

Figure 3: Only modes between μ1 and μ2 produce current

Notice that for each transverse mode, labeled by N, there is a dispersion relation E(N,k) with a cutoff ε = E(N,k=0) below which the mode cannot propagate, like waves in a waveguide. Hence, there are M propagating modes given by $M(E)=\sum_N \theta(E-\varepsilon_N)$ where θ is the step function.

Examining current for a particular mode,

$$
\begin{aligned}
I &= env \\
I &= \frac{e}{L}\sum_k \frac{1}{\hbar}\frac{dE}{dk} f(E) \\
I &= \frac{e}{L}\frac{2L}{2\pi}\int \frac{dk}{\hbar}\frac{dE}{dk} f(E) \\
I &= \frac{2e}{h}\int_{\varepsilon}^{\infty} f(E)dE
\end{aligned}
$$

where e = electron charge, n = electrons per length, v = electron velocity, and f(E) is electron distribution. The integral is from ε to infinity since ε is the lowest energy for that mode.

Now, for more than one mode, current is simply:

$$I = \frac{2e}{h}\int_{-\infty}^{\infty} f(E)M(E)dE$$

If M is constant over the bias μ_1 to μ_2,

$$
\begin{aligned}
I &= \frac{2e^2}{h}M\frac{(\mu_1-\mu_2)}{e} \\
I &= G_C V
\end{aligned}
$$

Therefore,

$$G_C = \frac{2e^2}{h}M$$

C. Step Response and Voltage Drop

We know M, assuming periodic boundary conditions, because the propagating k must be between $-k_f$ and k_f, where k_f is the Fermi momentum. Therefore, for width W, the separation between k values is $2\pi/W$, and therefore,

$$M = \frac{2 \cdot k_f}{2\pi / W}$$
$$M = \frac{k_f W}{\pi}$$
$$M = \frac{W}{\lambda_f / 2}$$

where λ_f = Fermi wavelength.

By decreasing the width of the conductor, the number of modes decreases, and we see the contact resistance drops in steps of $2e^2/h$ as seen in figure 4. For metals, λ_f is small; therefore, are many modes so when the width changes, there is a negligible change in current. On the other hand, λ_f can be quite large in semiconductors so we can see experimentally the step behaviour.

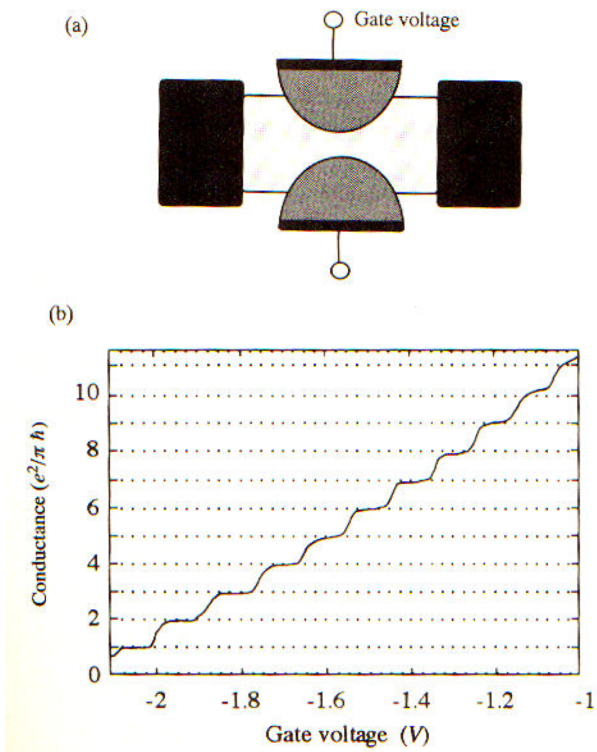

Figure 4: Conductance as a function of Gate Voltage (which controls the width of the conductor)

There is no voltage drop across the conductor, but there is a drop of $(\mu_1 - \mu_2)/e$ across the contacts. For the +k states, the +Fermi level, f^+, drops at the right contact while for the –k states, the -Fermi level, f^-, changes at the left contact. Let us define voltage to be the average Fermi level for the +k and –k states.

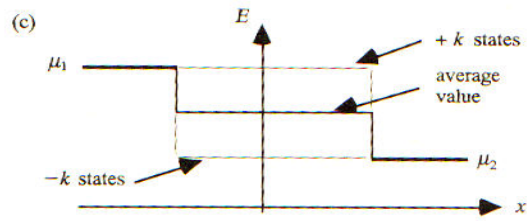

Figure 5: Voltage drops at the interface between the conductor and the contact

III. Landauer Formula

A. Proof of Landauer Formula

We now consider conductors where the transmission probability, T, is not unity. Let us guess that $G = 2e^2/h\ MT$, the Landauer formula, and show that we arrive at the correct conductance relationships for large and small conductors. Notice that this formula matches the one derived for the ballistic conductor (T = 1) and makes sense for T = 0; conductance is zero when there is no transmission.

Proof:

$$G = \frac{2e^2}{h} MT$$
$$G = \frac{2e^2}{h} \frac{k_f W}{\pi} T$$
$$G = \frac{2e^2}{2\pi\hbar} \frac{\sqrt{2\pi n_s} W}{\pi} T \frac{m\hbar}{m\hbar}$$
$$G = e^2 \frac{\sqrt{2\pi n_s} W}{\pi} T N_s \hbar$$
$$G = \frac{e^2 v_f W T N_s}{\pi}$$

where $k_f = \sqrt{2\pi n_s}$, where n_s = electron density, $N_s = m / \pi\hbar^2$, $v_f = \sqrt{2\pi n_s}\hbar / m$

$$G = \frac{e^2 v_f W N_s}{\pi} \frac{Lo}{L + Lo}$$
$$G = \frac{e^2 v_f W N_s}{\pi} \frac{Lo}{L + Lo}$$
$$G = e^2 W N_s \frac{D}{L + Lo}$$
$$G = W\sigma \frac{D}{L + Lo}$$

Using the Einstein relation by writing current density as $\overline{J} = en_s \overline{v_d}$, where $\overline{v_d}$ = drift velocity, and relating drift velocity to electric field then comparing $\overline{J} = \sigma \overline{E}$, we find $\sigma = e^2 N_s D$. In the following section, we will prove *T = Lo/L+Lo* where Lo is a constant.

Therefore,

$$G^{-1} = \frac{L + Lo}{W\sigma D}$$
$$G^{-1} = \frac{L}{W\sigma D} + \frac{Lo}{W\sigma D}$$
$$G^{-1} = G_s^{-1} + G_c^{-1}$$

where G_s^{-1} is the resistance from Ohm's law and G_c^{-1} is the contact resistance, and we arrive at the correct formula for conductance.

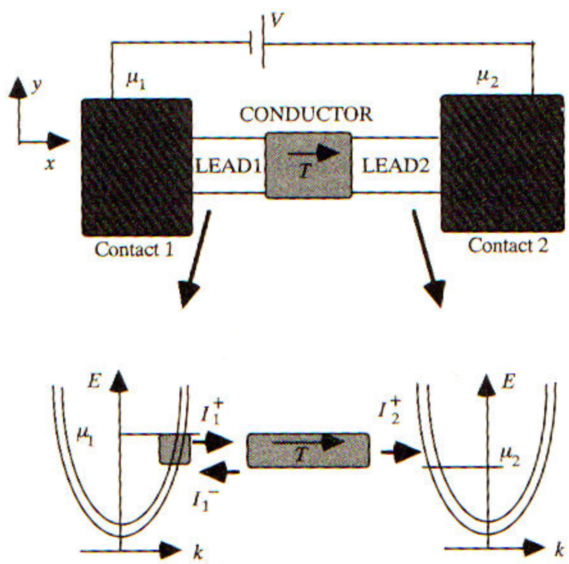

Figure 6: Setup similar to that for the ballistic conductor except there is a transmission probability

B. Derivation of T

Let us now consider two conductors in series, each with a scatterer. The probability of transmission through the first scatterer is T_1 and the probability of transmission through the second scatterer is T_2 (and reflection probabilities R_1 and R_2). The probability of transmission through both conductors T_{12} is not T_1T_2. If this was the case, for a chain of scatterers, the probability of passing through all the scatterers would decrease exponentially and we would not arrive at Ohm's law. There are an infinite ways of transmitting through the two conductors: directly T_1T_2, reflecting twice $T_1R1R_2T_2$, reflecting four times, $T_1R_1R_2R_1R_2T_2$, etc as shown in figure 7. Summing these probabilities, T_{12} = $T_1T_2/(1-R_1R_2)$

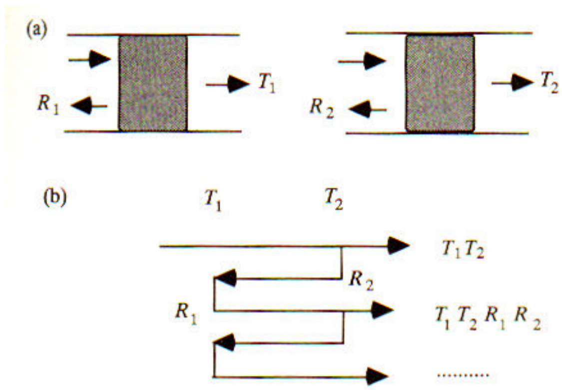

Figure 7: Transmission through two conductors, each with a scatterer

The expression $(1-T_{12})/T_{12} = (1-T_1/)/T_1+1-T_2/T_2$ has an additive property. Therefore, In general for N scatterers, $1-T(N)/T(N) = N(1-T)/T$ when we take the transmission probabilities for each scatterer to be the same. Therefore,

$$T(N) = \frac{T}{N(1-T)+T}$$

$$T(N) = \frac{T}{\nu L(1-T)+T}$$

$$T(N) = \frac{Lo}{L+Lo}$$

where $N=\upsilon L$ where υ is the linear density of the scatters.

C. Energy Distribution and Voltage Drop

We now consider the energy distribution of the electrons far away and near the scatterer. On the left of the scatterer, the +k states are have the same energy as the left contact μ_1. Similarly, on the right of the scatterer, the –k states have energy μ_2. On the right just after the scatterer, +k states from 0 to μ_1 are filled proportional to T. In addition, +k states are filled from the reflection of the –k states from 0 to μ_2 proportional to 1-T. Similarly for the –k state electrons near the scatterer on the left side but reversing the probabilities. Far from the scatterer on the right, the electrons in the +k states redistribute their energy to occupy the lower energy and reach a new highest energy level of F'' while the –k far left of the scatterer redistribute to F' as shown in figure 8. Notice that $F' = \mu_2+(1-T)(\mu_1-\mu_2)$ and $F'' = \mu_2+T(\mu_1-\mu_2)$.

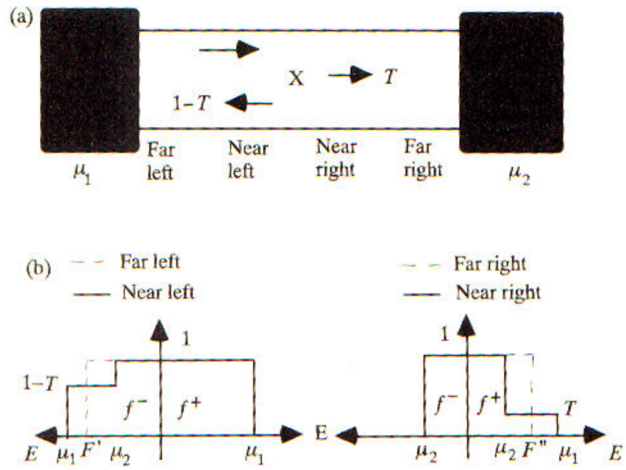

Figure 8: Energy distribution of electrons far and near scatterer

Therefore, one gets the following Fermi levels for the +k and –k states:

Left: $f^+(E) = \theta(\mu_1 - E)$
Near left: $f^-(E) = \theta(\mu_2 - E) + (1-T)\{\theta(\mu_1 - E) - \theta(\mu_2 - E)\}$
Far left: $f^-(E) = \theta(F' - E)$

Right: $f^-(E) = \theta(\mu_2 - E)$
Near right: $f^+(E) = \theta(\mu_2 - E) + T\{\theta(\mu_1 - E) - \theta(\mu_2 - E)\}$
Far right: $f^+(E) = \theta(F'' - E)$

If the energy of the +k electrons is $\mu 1$ before the scatterer and μ_2+T(μ_1-μ_2) after the scatterer, the potential difference across the scatterer is just the difference (1-T) (μ_1-μ_2)/e. We get the same result if we consider the –k states. Therefore, the resistance is at the scatterer. Notice that the remaining potential drop when compared to the original bias of μ_1-μ_2 is T(μ_1-μ_2) which is exactly the voltage drop across the contact.

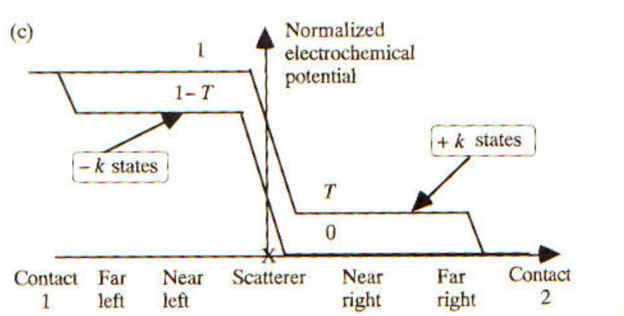

Figure 9: Voltage Drop across scatterer is (1-T)(μ_1-μ_2) and across contact is T(μ_1-μ_2)

D. Heat Dissipation

Scatterers are assumed to be rigid and have no internal degrees of freedom; therefore, they cannot dissipate heat. The dissipation of heat I^2 / G_S is from the evolution of energy distribution after the scatterer. Rewriting the equation for heat dissipation,

$$P_D = \frac{dI_U}{dz}$$

$$P_D = \frac{I}{e}\frac{dU}{dz}$$

where P_D = heat dissipation, energy current = $I_U = \frac{1}{e}\int Ei(E)dE$, average energy of the current = $I = \int i(E)dE$, U and I = net current.

$$U = \frac{eI_U}{I}$$

$$U = \frac{\int Ei(E)dE}{\int i(E)dE}$$

$$U = \frac{\int E(2eM/h)(f^+ - f^-)dE}{\int (2eM/h)(f^+ - f^-)dE}$$

Therefore,

$$U = \frac{F' + \mu 1}{2} \text{ on the far left}$$

$$U = \frac{\mu 1 + \mu 2}{2} \text{ near the scatterer}$$

$$U = \frac{F'' + \mu 2}{2} \text{ on the far right}$$

Since there is heat dissipation as long as the electrons have not yet reached equilibrium far from the scatterer because dU/dz is not zero, the dissipation is not only at the scatterer.

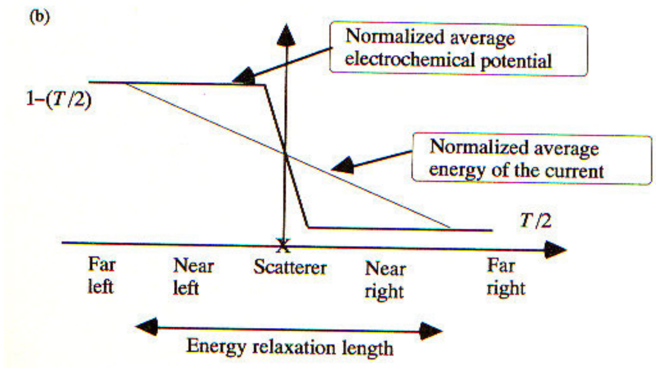

Figure 10: Heat dissipation (where the average energy of the current changes) is not at the scatterer

IV. References

Datta, Supriyo. Electronic Transport in Mesoscopic Systems. Cambridge, New York, (2003).