PhD(Astro) Qualifying examination - 2016

13:30 - 17:30, 2 September 2016

Do not open the exam until instructed to do so
but you may read this cover sheet

Instructions:

A one-page (8.5×11 inch) hand-written double-sided sheet of notes is allowed.

A scientific calculator is allowed and expected.

Put your name on upper right corner of your exam book.

All answers must be written in the exam books.

Start every question on a new page.

There are 8 questions to choose from. You will only select and provide answers for 5 of them; you may not attempt any portion of the other 3 questions. The five questions you choose all have equal value, and thus you may wish to time budget about 45 minutes per question.

On the front of your exam booklet you should clearly indicate which 5 questions you wish graded if you have any writing in your answers for more than 5 questions. If not, the first 5 questions started in your exam booklets will be graded.

Please return this examination with your exam booklet.

1. The Juno spacecraft arrived at Jupiter on July 4, 2016, to begin examination of the planet's gravity field and magnetosphere from an orbit that will bring it very close each passage. For this question you may assume the answer from each previous part even if you do not derive the previous answer. You must show your work clearly for full points. Some facts:

$$R_J = 71,500 \text{ km},$$
$$a_J = 5.2028870 \text{ AU},$$
$$GM_J = 1.26686 \times 10^8 \text{ km}^3/\text{s}^2,$$
$$GM_\odot = 1.32712438 \times 10^{11} \text{ km}^3/\text{s}^2.$$

(a) (5 points) Assume a lowest energy Hohmann transfer ellipse and show that Juno encountered Jupiter with a relative speed of $v_\infty \simeq 5.7$ km/s, taking the heliocentric orbits of Jupiter and Earth to be circular and in the same orbital plane.

(b) (5 points) The spacecraft arrived 4700 km above the cloud tops (radius given above) for its Jupiter Orbit Insertion (JOI) where it was at perijove (pericenter for an osculating orbit). Show that it was moving at 57.9 km/s relative to Jupiter.

(c) (6 points) The probe fired engines in a 35-minute burst that slowed its speed by 542 m/s. Approximating this as a single instantaneous impulsive burst, compute the semimajor axis if the new jovianocentric orbit, both in km and AU, the orbital eccentricity, and period in days. Explain in 1-3 sentences why this kind of orbit is needed for the scientific goals of the mission.

(d) (4 points) The Hill sphere (similar to the tidal Roche lobe for binaries) roughly gives the region of gravitational dominance of the planet. Orbits which take an object out more than half a Hill sphere away are very perturbed. First compute the jovian Hill sphere, and then decide if the probe's first orbit around Jupiter will enter this heavily perturbed regime. If you did not answer (c) then please use 0.050 AU as the Juno orbital semimajor axis.

Solutions:

(a) From Kepler and Newton,

$$v^2 = GM_\odot \left(\frac{2}{r} - \frac{1}{a} \right)$$

Juno's orbit has a perihelion distance of 1 AU and an aphelian distance of a_J. Thus, the semimajor axis of the orbit is $a = (a_J + 1 \text{ AU})/2 = 3.101$ AU, and at Jupiter, $r = a_J$. Therefore Juno's speed is

$$v = \left[GM_\odot \left(\frac{2}{a_J} - \frac{2}{a_J + 1 \text{ AU}} \right) \right]^{1/2} = \left[\frac{2GM_\odot \text{ AU}}{a_J(a_J + 1 \text{ AU})} \right]^{1/2} = 7.41 \text{ km/s}$$

Jupiter's orbital speed is

$$v_J = \left[\frac{GM_\odot}{a_J} \right]^{1/2} = 13.06 \text{ km/s}.$$

Jupiter is overtaking Juno, so the relative speed is $\Delta v = v_J - v = 5.65$ km/s.

(b) Apply the same equation to the orbit around Jupiter, letting $a \to \infty$. Add to this the kinetic energy that the probe already has as it arrives at Jupiter.

$$v^2 = \frac{2GM}{R_j + 4700 \text{ km}} + (\Delta v)^2.$$

which gives $v = 57.940$ km/s.

(c) Now use the Kepler equation to find the semimajor axis of the new orbit around Jupiter.

$$\frac{1}{a} = \frac{2}{r} - \frac{v^2}{GM_J} = \frac{2}{R_j + 4700 \text{ km}} - \frac{(57.940 - 0.542)^2}{GM_J}$$

which gives $a = 4145286$ km $= 0.0277$ AU.
The pericenter distance is $r_p = a(1 - \epsilon)$, so the eccentricity is

$$\epsilon = 1 - \frac{R_j + 4700 \text{ km}}{4145286 \text{ km}} = 0.9816.$$

The period is given by Kepler's third law,

$$P = \left(\frac{4\pi^2 a^3}{GM_J}\right)^{1/2} = 4711371 \text{ s} = 54.53 \text{ day}.$$

A very deep pericenter and large range of distances is the only way to have sensitivity to the higher-order multipoles of both the gravity field (J_2, J_4, etc) and magnetosphere. This remote sensing is how interior models of the planet will be constrained.

(d) The radius of Jupiter's Hill sphere is

$$r \simeq a_J \left(\frac{M_J}{3M_\odot}\right)^{1/3} \simeq 0.355 \text{ AU}.$$

So the orbit will not enter the heavily-perturbed region.

2. (a) (5 points) Write down an equation for the power emitted in synchrotron radiation by a single electron moving in a magnetic field, in terms of the energy density of the magnetic field and the Lorentz factor of the electron.

   (b) (5 points) The number density of the electrons is n_e and they fill a spherical region of radius R. Estimate the energy density of photons within the sphere, assuming that it is optically thin.

   (c) (5 points) What is the inverse Compton emission from a single electron passing through a gas of photons field in terms of the energy density of the photons and the Lorentz factor of the electron?

   (d) (5 points) What is the total inverse Compton emission from the region if you assume that the synchrotron emission provides the seed photons for the inverse Compton emission?

Solutions:

(a)

$$P_B = \frac{4}{3}\gamma^2 c\beta^2 \sigma_T U_B$$

(b) Pn_e gives the power per unit volume. To get the energy per unit volume we have to multiply by the typical time for photons to escape the spherical region typically R/c because it is optically thin so we have

$$U_{\text{photon}} = \frac{4}{3}\gamma^2 \sigma_T c\beta^2 U_B n_e \frac{R}{c}$$

(c)

$$P_{\text{IC}} = \frac{4}{3}\gamma^2 c\beta^2 \sigma_T U_{\text{photon}}$$

(d)

$$P_{\text{IC}} = \frac{4}{3}\gamma^2 c\beta^2 \sigma_T \left(\frac{4}{3}\gamma^2 c\beta^2 \sigma_T U_B n_e \frac{R}{c}\right) n_e V$$

so

$$P_{\text{IC}} = \frac{64}{27}\pi\gamma^4\beta^4 c\sigma_T^2 U_B n_e^2 R^4$$

3. (a) (5 points) Show that for a gaseous protoplanetary disc in equilibrium, the gas orbital speed must be less than the Keplerian circular speed in regions that have an outward decreasing pressure gradient. For simplicity, consider only the midplane of the disc. Define your terms.

   (b) (5 points) Now consider the evolution of a cm-sized particle in a gaseous protoplanetary disc. As before, let the disc pressure decrease with increasing radius. However, now suppose that in a very localized radial region of the disc, the radial pressure gradient briefly becomes positive before returning to its normal negative profile. Describe the orbital evolution of the cm-sized particle if it were originally exterior to the local pressure gradient inversion. Be succinct and clear.

   (c) (10 points) What is the direction and speed of a cm-sized particle if it is placed at 1 AU in a solar nebula-like disc around a solar mass star? Let the pressure gradient decrease with increasing radius over the entire region of interest. Use power-law parameterizations for the midplane gas density and temperature along with the ideal gas law. Specifically, let $T(R) = 300(R/\mathrm{AU})^{-m}$ K and $\rho(R) = 10^{-9}(R/\mathrm{AU})^{-n}$ g/cc, where $m = 0.5$ and $n = 2.5$. For further simplicity, you may use the Epstein regime for calculating the particle's stopping time $t_s = \rho_s s/(\rho c)$ in the gaseous disc. Here, ρ_s is the internal density of the cm-sized solid, s is the solid size, ρ is the gas density, and c is the sound speed. Ignore turbulence. Hint: estimate the radial drift by $\Delta g t_s$, where Δg is the "residual gravity", i.e., the radial force imbalance caused by the particle orbiting with the gas instead of the Keplerian speed. Make reasonable assumptions for any additional values that are needed.

Solutions:

(a) Hydrostatic equilibrium demands that

$$\frac{1}{\rho}\frac{dP}{dR} = -\Omega_k^2 R + \Omega^2 R,$$

where ρ is the gas density, P is the pressure, R is radial distance from the star, $\Omega_k = \sqrt{(GM/R^3)}$ is the Keplerian circular angular speed (mean motion), and Ω is the gas orbital angular speed. Rearranging the terms we see that

$$\Omega^2 = \Omega_k^2 + \frac{1}{\rho R}\frac{dP}{dR},$$

which means that if $dP/dR < 0$, then the orbital speed of the gas must be less than the Keplerian speed. The solution may be written using either the angular or linear speed.

(b) A negative radial pressure gradient would cause the gas flow to be sub-Keplerian. Because the cm-sized particle is not supported by hydrostatic pressure, it must orbit at the Keplerian speed to maintain orbit. The difference in the orbital speeds would result in the particle feeling a headwind, which would cause it to lose angular momentum due to drag and drift inward. However, a positive pressure gradient demands that the gas orbital speed be super-Keplerian. Thus, if the particle were placed in the positive gradient region, it would gain angular momentum due to drag and drift outward. Altogether, if the cm-sized particle were to start exterior to the pressure inversion, the cm-sized particle would drift inward until it reached the inversion, and then would become trapped where the pressure gradient transitions from positive to negative.

(c) As given above, the force imbalance is just due to the pressure gradient term in hydrostatic equilibrium,

$$\Delta g = \frac{1}{\rho} dP/dR.$$

Let $T = T_0(R/R_0)^{-m}$ and $\rho = \rho_0(R/R_0)^{-n}$. This means $dP/dR = -(n+m)P/R$ for our assumptions. Using the sound speed c and adiabatic index γ,

$$dP/dR = -(n+m)\rho c^2/(\gamma R),$$

so

$$\Delta g = -(n+m)c^2/(\gamma R).$$

Using the definition of the stopping time, the radial drift speed is

$$v_r = -(n+m)\rho_s s c/(\gamma R \rho).$$

For $n = 2.5$, $m = 0.5$, $\gamma = 1.4$, $c = 1.2$ km/s, $s = 1$ cm, $\rho = 1 \times 10^{-9}$ g/cc, $\rho_s = 3$ g/cc, $R = 1.5 \times 10^{13}$ cm, $v_r = -50$ cm/s (drifting inwards).

Notes: 1) accept the isothermal sound speed instead of the adiabatic should they use it, which would negate the use of γ. 2) The sound speed is derived assuming a temperature of 300 K, an adiabatic index of 1.4, and a mean weight of 2.3 g/mol.

4. Using the concept of virial equilibrium, gravitational potential energy and kinetic energy due to thermal motion and degeneracy, calculate the maximum temperature that can be achieved by a spherical mass of hydrogen gas of uniform density.

   Solution:

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

$$K = \frac{M}{m_p}\left(3kT + \frac{p_F^2}{2m_e}\right)$$

   Now

$$\frac{8\pi}{3}\frac{p_F^3}{h^3} = n = \frac{M}{m_p}\frac{3}{4\pi R^3}$$

   so

$$p_F^3 = \frac{M}{m_p}\frac{3}{4\pi R^3}\frac{3h^3}{8\pi}$$

   Set $U + 2K = 0$ and solve for $T = T(R, M)$. Take dT/dR and find the maximum temperature for a given mass.

$$kT = \frac{1}{5}\frac{GM^2}{R}\frac{m_p}{M} - \frac{p_F^2}{6m_e}$$

   where p_F is given in terms of M and R above so

$$kT = \frac{GMm_p}{5R} - \frac{h^2}{6m_eR^2}\left(\frac{9M}{32m_p\pi^2}\right)^{2/3}$$

   and

$$\frac{d(kT)}{dR} = -\frac{GMm_p}{5R^2} + 2\frac{h^2}{6m_eR^3}\left(\frac{9M}{32m_p\pi^2}\right)^{2/3}$$

   and

$$kT_{max} = \frac{64^{2/3}G^2M^{4/3}m_em_p^{8/3}\pi^{4/3}}{259^{2/3}h^2}$$

   This is about 40 MK for one solar mass or 3.6 keV. Just about right!

   For 100 solar masses you get 1.6 MeV so unless you have lots of nuclear burning ahead of time you will get electron-positron pairs in such a massive star (pair-instability SN). On the other hand for 0.1 solar mass you only get 2 MK, barely enough for nuclear burning.

5. (20 points) Currently, the best evidence for the direct detection of dark matter comes from the DAMA experiment that looks for flashes of light produced when dark matter particles interact with a sodium iodide crystal located deep underground in Italy's Gran Sasso laboratory. The DAMA team reports a 9-sigma signal that peaks in the summer and wanes in the winter. This is expected theoretically as in summer the Earth is traveling in the same direction as the Sun through the galactic dark matter halo and therefore encounters dark matter particles more frequently.

   Assuming that the dark matter particles have a thermal velocity distribution, predict the magnitude of the summer-winter peak-to-peak variation in the detection rate if these events are due to dark matter interactions. The period of the Sun's orbit around the is about 250 million years and the distance to the galactic centre is about 8.5 kpc.

   Solution:

   Let the velocity of the Earth with respect to the dark matter halo be V. Let the velocity component of a dark matter particle in this same direction be v. The relative velocity is then

$$\Delta v = V - v$$

   The collision rate is

$$R = n\sigma\Delta v$$

   where σ is the collision cross section and n is the number density of dark matter particles.

   Integrating this over the 1-d velocity distribution $f(v) \propto \exp(-mv^2/2kT)$ of the dark matter,

$$R = n\sigma \int_{-\infty}^{\infty} (V - v) f(v) dv = n\sigma V.$$

   The last equality follows because $f(v)$ is a symmetric function whose integral is unity.

   The ratio of rates is therefore

$$\frac{R_{\text{summer}}}{R_{\text{winter}}} = \frac{n\sigma(v_\odot + v_\oplus)}{n\sigma(v_\odot - v_\oplus)} = \frac{1+x}{1-x}$$

   where $x = v_\oplus / v_\odot$ is the ratio of the Earth's orbital velocity about the Sun to the Sun's orbital velocity about the galactic centre. These are, respectively,

$$v_\oplus = 2\pi \text{ AU}/1 \text{ yr},$$
$$v_\odot = 2\pi \times 8.5 \text{ kpc}/250 \times 10^6 \text{ yr}$$

   Therefore,

$$x = \frac{250 \times 10^6}{8500 \times 206265} = 0.1426.$$

   Thus the expected ratio of peak summer/winter rates is

$$\frac{R_{\text{summer}}}{R_{\text{winter}}} = \frac{1+x}{1-x} = 1.33$$

   So we expect a 33% variation.

6. (a) (10 points) Show that, for a homogeneous universe filled with a perfect gas of density ρ and pressure P, expanding adiabatically, the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}$$

follows from the first Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3},$$

and the first law of thermodynamics

$$dU = TdS + PdV$$

Here a is the scale factor, k is the curvature constant, Λ is the cosmological constant, U is the energy contained within some volume V, and we are using units in which $c = 1$.

(b) (10 points) Assume a zero cosmological constant and a matter-dominated flat universe where eventually the pressure becomes negligible. At large times how does the scale factor depend on time t?

Solutions:

(a) In an adiabatic expansion there is no change in entropy, so $dS = 0$. The first law becomes

$$d(\rho V) = V d\rho + \rho dV = PdV.$$

Therefore, since $V \propto a^3$,

$$d\rho = -(\rho - P)\frac{dV}{V} = -3(\rho - P)\frac{da}{a}$$

Now multiply the first Friedmann equation by a^2 and differentiate,

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\frac{d}{dt}\left(\rho a^2\right) + \frac{2}{3}\Lambda a\dot{a},$$

Evaluating the first term on the RHS and substituting from the first law,

$$\begin{aligned}\dot{a}\ddot{a} &= \frac{4\pi G}{3}\left[-3(\rho - P)a\dot{a} + 2\rho a\dot{a}\right] + \frac{1}{3}\Lambda a\dot{a},\\ &= -\frac{4\pi G}{3}(\rho + 3P)\,a\dot{a} + \frac{\Lambda}{3}a\dot{a}\end{aligned}$$

from which the desired result immediately follows.

(b) Set $\Lambda = 0$ and $k = 0$ in the first Friedmann equation. If there is no pressure, the first law of thermodynamics becomes $d(\rho V) = 0$ which requires $\rho \propto V^{-1} \propto a^{-3}$. Therefore, $\dot{a} \propto a^{-1/2}$. Integrating this with the boundary condition $a(0) = 0$ gives the solution $a(t) \propto t^{2/3}$.

7. Statistical inference

Suppose we have position measurements of galaxies in a small cluster and would like to know if the Elliptical and S0 galaxies are distributed differently. So we estimate the position of the cluster centre and compute the angular distances of each galaxy from this point. The resulting distances (in arcmin) for 18 galaxies are shown in the table below.

E	S0
3.8	12.5
8.6	5.3
2.1	7.8
4.9	11.2
6.1	9.8
3.3	6.7
4.4	8.3
10.2	2.3
0.4	
2.9	

Use the two-sample Kolmogorov-Smirnov test to estimate the probability that these two samples are drawn from the same population. Are the differences statistically significant?

For reference, the probability of finding a value of $\sqrt{mn/(m+n)}\,\sup|F-G|$ as large as x, if the two samples are drawn from the same parent distribution, is given by $1-K(x)$ where $K(x)$ is the Kolmogorov distribution

$$K(x) = 1 - 2\left[e^{-2x^2} - e^{-2\cdot 4x^2} + e^{-2\cdot 9x^2} - e^{-2\cdot 16x^2} + \cdots\right]$$

Here F and G are the cumulative distributions of the two samples and m and n are the sizes of the two samples.

Solution:

First sort the numbers for the two samples and place them in ascending order. Compute the cumulative distributions and the differences:

x_E	x_{S0}	$F(x_E)$	$G(x_{S0})$	diff
0.4		0.1	0.000	0.1
2.1		0.2	0.000	0.2
	2.3	0.2	0.125	0.075
2.9		0.3	0.125	0.175
3.3		0.4	0.125	0.275
3.8		0.5	0.125	0.375
4.4		0.6	0.125	0.475
4.9		0.7	0.125	0.575
	5.3	0.7	0.250	0.450
6.1		0.8	0.250	0.550
	6.7	0.8	0.375	0.425
	7.8	0.8	0.500	0.300
	8.3	0.8	0.625	0.175
8.6		0.9	0.625	0.275
	9.8	0.9	0.750	0.150
10.2		1.0	0.750	0.250
	11.2	1.0	0.875	0.125
	12.5	1.0	1.0	0.000

The largest difference is 0.575 which gives

$$x = \sqrt{\frac{10 \cdot 8}{10 + 8}} \cdot 0.575 = 1.212$$

Evaluating the first few terms in the series,

$$1 - K(x) = 2(0.052924 - 1.84 \times 10^{-6} + \cdots) \simeq 0.1058$$

So this difference is not significant (it has a $\sim 10\%$ chance probability).

8. Observational Astronomy

(a) (10 points) You are planning to do UBV photometry of stars in galactic centre ($\alpha = 17^h\ 45^m\ 40.04^s, \delta = -29^\circ\ 00'\ 28.1")$ using the Gemini North telescope on Mauna Kea (155° 28′ 05"W, +19° 49′ 14"N). What is the optimal time of year for this observation, and what will be the minimum possible airmass?

(b) (5 points) The UBV extinction coefficients for Mauna Kea are $k_{\rm U} = 0.37$, $k_{\rm B} = 0.17$, and $k_{\rm V} = 0.12$. Roughly how large a correction would be needed to remove the atmospheric extinction from the measured U-B and B-V colours?

(c) (5 points) The median atmospheric seeing at Gemini North is 0.64 arcsec FWHM (at zenith for a wavelength of 0.5 um). Do you expect that the typical image quality would be better or worse than this for your galactic-centre observations? Would the image quality be better in U or in V?

Solutions:

(a) Ideally, we would like the CG to cross the meridian (where is is highest in the sky) around midnight. The sidereal time $s = \alpha + h$, where h is the hour angle. On the meridian, $h = 0$, so the sidereal time would be about 17.5 hrs. On the vernal equinox, typically March 20, the Sun is at $\alpha = 0$, so the sidereal time at midnight is 12 hr on that date, and increases by two hours each month. So the best time to observe the GC will be $(17.5 - 12)/2 = 2.75$ months after Mar 20, so mid June would be best. On the meridian, the GC will have a zenith angle of $\zeta = 19.82^\circ + 29.00^\circ = 48.82^\circ$. The airmass will be $X = \sec\zeta = 1.52$.

(b) The magnitude correction in each band is $\Delta m = -k_m X$. For $X = 1.52$ the (U,B,V) corrections will be $(-0.562, -0.258, -0.018)$ so the corrections to the colours would be approximately $\Delta(U - B) = -0.304$ and $\Delta(B - V) = -0.240$.

(c) Seeing is proportional to $\lambda^{-1/5}$ and $X^{3/5}$. Therefore, the following median values are expected at $X = 1.52$:

Band	Wavelength (um)	seeing (arcsec)
U	0.36	0.88
B	0.44	0.84
V	0.55	0.80

Full marks for saying that the image quality will be worse than 0.64 arcsec and that it is worse in U than V.