Autumn 2017 Physics & Astronomy Qualifying Exam
for Advancement to Candidacy
Day 1: August 31, 2017

Do not write your name on the exam. Instead, write your student number on each exam booklet. This will allow us to grade the exams anonymously. We'll match your name with your student number after we finish grading. If you use extra exam booklets, write your student number on the extra exam books as well. Write all answers in the blank exam booklet(s), not on this printout!

Today's portion of the exam has 8 questions. Answer *any five* of the eight. Do not submit answers to more than 5 questions—if you do, only the first 5 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 4 hours to complete 5 questions.

You are allowed to use one $8.5'' \times 11''$ formula sheet (both sides), and a handheld, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	-273°C
atomic mass unit	1 amu	1.66×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
distance from earth to sun	1 AU	1.5×10^{11} m
Laplacian in spherical coordinates	$\nabla^2 f =$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
mass of an electron	m_e	0.511 MeV/c^2
mass of the sun	M_{sun}	2×10^{30} kg
mass of a proton	m_p	938 MeV/c^2
mass of a neutron	m_n	940 MeV/c^2
mass of hydrogen atom	m_H	1.673×10^{-27} kg
molecular weight of H_2O		18
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
nuclear magneton	μ_N	5×10^{-27} J/T
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the Earth	R_{earth}	6.4×10^6 m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Stirling's approximation	$N!$	$e^{-N}N^N\sqrt{2\pi N}$
Thomson cross section	σ_T	6.65×10^{-29} m^2

1. There is a theoretical limit to the luminosity that an object bound gravitationally can emit. This is called the Eddington luminosity L_{edd} and it is reached when the inward gravitational force on a piece of material is balanced by the outgoing radiation pressure. Consider a cloud of hydrogen surrounding a central object of mass M and luminosity L. Let σ_T be the Thomson cross-section (the cross-section for elastic scattering of photons on hydrogen) and m_H be the mass of a proton. Show that the maximum luminosity L_{edd} that the central object can have and still not expel the cloud of hydrogen by radiation pressure is given by

$$L_{edd} = 4\pi c G M m_H / \sigma_T$$

Calculate L_{edd} for the Sun.

2.

A. A quantum mechanical particle of mass m, charge q, and zero spin moves in two dimensions in a potential given, as a function of the polar coordinates r and θ, by $V(r, \theta) = \frac{1}{2}kr^2 + \frac{3}{2}kr^2 \sin^2 \theta$. Calculate the energies and degeneracies of the lowest 6 energy levels.

B. Now an electric field E is added along the $\theta = 45^\circ$ direction. Calculate the resulting shift in the energy levels from part A.

3. “Tumour treating fields” is a novel medical treatment for brain cancer that relies on applying low-intensity alternating electrical fields to the brain. It is hypothesized that this therapy may act on tubulin protein assemblies, which have a permanent electric dipole moment, by preventing them from orienting properly during cell division and so slowing tumour growth. The treatment is only effective if the frequency of the applied field is lower than the lowest frequency at which the tubulin molecules rotate, since otherwise the field will average to zero during the time it takes the dipole moment to reorient itself.

The tubulin complex has a mass of 110,000 amu, an electric dipole moment of 40 $e \cdot$nm, and dimensions of $4.6 \times 8.0 \times 6.5$ nm. Do an order of magnitude calculation of the approximate maximum frequency at which the electric field should be applied.

4. A reversible heat engine is built using a photon gas as the working medium. At Point 1, the temperature of the gas and surrounding chamber is T_h and its volume is zero. The gas is slowly expanded isothermally until it reaches volume V_2. It then expands adiabatically to volume V_3, at which point its temperature is T_c. Next it contracts isothermally to volume zero. Finally the system is heated at constant volume to return it to Point 1.

Draw a diagram of of this cycle in the PV plane. Then calculate the work done, heat absorbed, and change in entropy of the gas during each of the four steps of the cycle.

Hint: the internal energy of a photon gas is $U = bVT^4$, where b is a constant, and for any ultra-relativistic gas $U = 3PV$.

5.

A. A quantum particle in 1D is prepared in an state localized at $x = 0$, with an initial position distribution given by a Gaussian of RMS width σ. The potential is $V(x) = 0$ everywhere. Estimate the probability density of such a particle at a later time t as a function of x.

B. The particle is then prepared in an equal amplitude superposition of two spatially strongly localized states with widths $\sigma \ll d$ peaked at $-d/2$ and $+d/2$ at time $t = 0$. Estimate the probability density of such a particle at a later time t, and comment explicitly on any interference effects that are present.

6. A tall, thin, vertical chimney of length h begins to topple over, as a rigid unit, by pivoting from its base. When it has tipped by some angle θ_{break}, it breaks, due to the internal torque exceeding the strength of the material which is constant along its length. Determine the position x along the chimney, as measured from its base, where it will break.

7. Two oscillating dipole moments p_1 and p_2 are positioned on the x-axis at positions $x = \pm L/2$, and are oriented in the $+z$-direction. They oscillate in phase and at the same angular frequency ω. Consider the radiation emitted in the xz plane at polar angle θ relative to the z-axis.

Note that the electric field for a single radiating dipole in the far zone is proportional to

$$\vec{E} \propto [(\hat{n} \times \vec{p}) \times \hat{n}] \frac{e^{ikr}}{r}$$

where r is the distance from the dipole and $\hat{n}$ is the normal vector pointing from the dipole to the measurement position.

A. Find an expression for $\vec{E}_{rad}$ in the far zone.

B. Use this to show that the differential radiated power obeys

$$\frac{dP}{d\Omega} \propto \sin^2 \theta (p_1^2 + 2p_1 p_2 \cos \delta + p_2^2)$$

and determine the value of the parameter δ.

C. Show that when L is much smaller than the wavelength of the radiation, the radiation is the same as from a single oscillating dipole of amplitude $p_1 + p_2$.

8. In an NMR apparatus, a magnetic field of 10 T along the z-axis is used to (partially) polarize the hydrogen nuclear spins in a sample of room temperature water that is 1×1×1 cm^3. The spins are quickly rotated into the x-y plane using an AC field along the y-axis, and then the AC field is turned off and the nuclear spins precess around the primary 10 T field. A 1000 turn coil with radius 2 cm, with its axis also pointing along the y axis, surrounds the water sample. Estimate the amplitude of the AC voltage induced in the coil due to the precessing spins. Note that the proton's magnetic moment is 1.4×10^{-26} J/T, and that the magnetic field of a dipole is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \left(\frac{3\vec{r}(\vec{m} \cdot \vec{r})}{r^5} - \frac{\vec{m}}{r^3} \right)$$

Autumn 2017 Physics Qualifying Exam
for Advancement to Candidacy
Day 2: September 1, 2017

If you are in the Ph.D. in astronomy program, don't write this exam! Ask a proctor for the astronomy version of today's exam!

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9. The wavefunction of the ground state of the hydrogen atom is

$$\Psi(r,\theta,\phi) = \frac{1}{\sqrt{4\pi}}\frac{2}{a_0^{3/2}}e^{-r/a_0}$$

where a_0 is the Bohr radius. This wavefunction was calculated in the approximation that the proton is a point particle. Suppose instead that the proton were a sphere with uniform charge density and radius $R \ll a_0$. How much would the binding energy of the ground state shift compared to the point particle case? You may leave your answer in the form of an integral.

10. Consider two spatially localized spin 1/2 particles, coupled by an exchange interaction, and immersed in an inhomogeneous magnetic field. They can be described by the Hamiltonian

$$H = -J\left[S^1_+ S^2_- + S^1_- S^2_+\right] - h_1 S^1_z - h_2 S^2_z$$

where the spin operators are normalized by

$$\left[S^a_i, S^b_j\right] = i\delta^{ab}\epsilon_{ijk}S^a_k$$

In this expression S^a_i is the spin operator for spin a in the i^{th} Cartesian direction.

Find all of the eigenvalues of H.

Useful facts:

$$S_\pm = S_x \pm iS_y$$

$$S_x = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], \quad S_y = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right], \quad S_z = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]$$

11. The classical equation of motion of a system of identical particles with positions $\vec{x}_1, ...\vec{x}_N$ interacting with a two-body potential is

$$m\ddot{\vec{x}}_a(t) = -\vec{\nabla}_a \sum_{b\neq a} V(|\vec{x}_a - \vec{x}_b|)$$

This equation is clearly invariant under the boost transformation of Galilean relativity,

$$\vec{x}_a(t) \ \rightarrow \ \vec{x}_a(t) + \vec{v}t, \quad \forall a = 1, \ldots, N$$

This allows us to compare physics in different reference frames.

The quantum mechanical free particle is described by a wave-function $\psi(\vec{x}_1, ..., \vec{x}_N, t)$ which obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x}_1, ..., \vec{x}_N, t) = \left[-\sum_{a=1}^{N}\frac{\hbar^2\vec{\nabla}^2}{2m} + \frac{1}{2}\sum_{a\neq b} V(|\vec{x}_a - \vec{x}_b|)\right]\psi(\vec{x}_1, ..., \vec{x}_N, t)$$

A. Calculate the classical energy of the boosted system in terms of m, $\vec{v}$, and the energy E_0 measured in the centre of mass frame.

B. Let $\psi_{CM}(\vec{x}_1, ..., \vec{x}_N, t)$ be the wavefunction in the centre of mass frame. The wavefunction in a boosted frame, from the point of view of an observer moving with constant velocity $\vec{v}$, can be shown to have the form $\psi_{boost} = \psi_{CM}\Phi(\vec{X}_{CM}, t)$, where $\vec{X}_{CM}$ is the position of the centre of mass. Show that ψ_{boost} satisfies the Schödinger equation and calculate $\Phi(\vec{X}_{CM}, t)$.

Hint: use your result from Part A to figure out the form of the kinetic energy operator in the boosted frame, expressed in terms of $\vec{x}_i$ and $\vec{X}_{CM}$.

12. Consider a relativistic elastic collision between a photon with initial energy E_i and an electron at rest. As a result of the collision the photon is now moving at an angle θ relative to its initial direction. Calculate the final energy of the photon.

13. Consider a cylindrical container of volume V separated into two volumes, v_1 and $v_2 = V - v_1$, by a freely movable partition. The left volume v_1 contains N monatomic molecules of mass m_L and the right volume v_2 contains M monatomic molecules of mass m_R. The partition allows heat to flow between the two volumes. The molecules are dilute enough that you can ignore interactions and the system can be treated classically.

A. Assume that the system is in contact with a heat bath at temperature T, and that the external walls conduct heat. Compute the entropy of the system as a function of v_1. What is the equilibrium value of v_1?

B. Now assume that the external walls but not the partition are completely heat-insulating and that the total energy in the cylinder is E. What is the equilibrium value of v_1 in this case?

C. If the mass of the partition is 0.1 kg and the temperature is 300 K, what is the mean square velocity of the partition in equilibrium?

14. When the universe was very young, protons and neutrons could be converted into each other by these reactions:

$$n + e^+ \rightarrow \bar{\nu}_e + p$$

$$n + \nu_e \rightarrow p + e^- .$$

At times much earlier than one second after the Big Bang, these reactions were fast and maintained the $n : p$ ratio at close to $1 : 1$. As the universe cooled, the equilibrium shifted to favour protons due to their lower mass. The rate of these reactions dropped precipitously when the universe reached a temperature of 8×10^9 K, causing both reactions to effectively cease. Over the next few minutes, about 20% of the neutrons decayed ($n \rightarrow p + e^- + \bar{\nu}_e$), while the rest were used to form helium-4 atoms. Estimate the ratio of the number of hydrogen nuclei to helium nuclei in the universe at the end of this process.

15. The Breakthrough Starshot project proposes to use powerful lasers to accelerate a lightweight space probe to a final velocity of $0.2c$. Suppose the probe has a mass of 10 g and has a 99% reflective solar sail with an area of 16 m^2, and that the laser shines on it for 10 minutes. Calculate the total amount of energy that strikes the sail during the acceleration, assuming that the spacecraft starts at rest.

16. Find an approximate expression for the terminal velocity of a cylindrical magnet, with diameter D and length also D, mass M, and magnetic moment μ falling down the center of a very long vertically-oriented copper tube (resistivity of copper is ρ) with radius R and wall thickness T. Make the following assumptions:

A. the axis of the magnet stays aligned with the (vertical) axis of the tube

B. $D \ll R$ and $T \ll R$

C. the terminal velocity is slow enough that self-inductance of the tube can be ignored, as can air resistance

Note that the magnetic field of a dipole is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}(\vec{m}\cdot\vec{r})}{r^5} - \frac{\vec{m}}{r^3}\right)$$

Hint: it's OK to express your answer as a function of M, μ, R, and ρ times a unitless definite integral. You don't need to evaluate the numerical value of that integral.