September 2011 Physics & Astronomy Qualifying Exam
for Advancement to Candidacy
Day 1: September 1, 2011

Do not write your name on the exam. Instead, sign up on the sign-up sheet, and note the serial number next to your name. Write that number on your exam in place of your name. This will allow us to grade the exams anonymously. We'll match your exam with your name using the sign-up sheet after we finish grading. If you use extra exam booklets, write your serial number on the extra exam books as well.

Today's portion of the exam has 8 questions. Answer *any five* of the eight. Do not submit answers to more than 5 questions—if you do, only the first 5 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 4 hours to complete 5 questions.

You are allowed to use one $8'' \times 11''$ formula sheet (both sides), and a hand-held, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	-273°C
air pressure at sea level	1 atm	10^5 N/m^2
atomic mass unit	1 amu	1.66×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
distance from earth to moon	d_{moon}	3.8×10^8 m
distance from earth to sun	1 AU	1.5×10^{11} m
electron volt	1 eV	1.6×10^{-19} J
mass of an electron	m_e	0.511 MeV/c^2
mass of the moon	M_{moon}	7×10^{22} kg
mass of the sun	M_{sun}	2×10^{30} kg
mass of a proton	m_p	938 MeV/c^2
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the Sun	R_{sun}	7×10^8 m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}

1. Consider a triangular molecule consisting of 3 protons fixed in space and a total of 2 electrons. The electron wave functions correspond to that of a 1s electron in atomic H when on any given proton modified by a tunneling matrix element t between the neighboring protons.

A. Calculate the eigenenergies (in units of t) and eigenstates for the case where there is no Coulomb repulsion between the electrons, and determine the lowest energy singlet ($\vec{J} = 0$) and triplet ($\vec{J} = 1$) state energies. (Ignore proton spin.)

B. Now turn on an infinite Coulomb repulsion if the two electrons are on the same proton site, and call this U. Since the electrons are fermions and with U infinite the electrons actually never "see" each other. Nonetheless there is a large splitting between the triplet and singlet states. Confirm this assertion and describe physically in terms of the rules of exchanging two fermions why this is the case.

2. (Part a of this problem has no relation to parts b or c).

(a) Can I get to the Andromeda Galaxy and back within my lifetime? The distance to this galaxy is 2.4×10^{19} km.

(b) A sound wave is a travelling adiabatic compression in a material. Given that the speed of sound c in a material satisfies $c^2 = \partial P/\partial \rho$, derive the speed of sound in an ideal diatomic gas at temperature T.

(c) The speed of sound dictates how fast pressure can act to resist perturbations in the density of a gas. By comparing the speed of sound to the speed at which gas would collapse under gravity if pressure didn't exist, derive the maximum mass of an ideal diatomic gas of density ρ at temperature T that is stable against gravitational collapse.

3. Ideal gas

A cylinder is partitioned by a membrane into a volume V_1 initially filled with a classical ideal gas of N particles with no internal degrees of freedom at temperature T, and a volume V_2 initially enclosing a perfect vacuum.

- The cylinder is in contact with a heat reservoir at temperature T. The membrane is moved slowly without friction, allowing the gas to fill the entire cylinder. Compute the work done by the gas, the heat transferred between the gas and the heat bath, and the change in the entropy of the gas. Is this a reversible process?
- The cylinder is returned to its initial state and insulated from the heat bath. The membrane is allowed to break, releasing the gas to fill the entire volume. Assume that the expansion occurs essentially instantaneously, and a new equilibrium is reached. Compute the work done by the gas and the change in the entropy. Is this a reversible process?

4. Galactic cockroaches

The Milky Way galaxy contains 100 billion stars, and has a radius of 250,000 light years. Suppose that humans set a long-term goal to colonize every inhabitable star system in the Milky Way, and that for argument's sake 10% of star systems are suitable. This colonization might be accomplished by sending out a ship of settlers to a nearby solar system, establishing a colony, and then waiting long enough (500 years?) for the colony to develop enough that it can send its own settler ships. Do an order of magnitude estimate of how long it would take to colonize the galaxy, explaining any assumptions you make. (Attention Star Trek fans: any solutions that violate known laws of physics will be given zero points.)

5. Assume that the eigenstates of a hydrogen atom isolated in space are all known and designated as usual by $\Psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_{lm}(\theta, \phi)$. Supppose the nucleus of the hydrogen atom is located at a distance d from an infinite potential wall which, of course, tends to distort the hydrogen atom.

A. Find the explicit functional form of the ground state wave function of this hydrogen atom in the limit that d approaches zero. (Don't bother to normalize it.)

B. Find all other eigenstates of this hydrogen atom in half-space, i.e. $d \to 0$, in terms of the usual isolated hydrogen atom eigenstates R_{nl} and Y_{lm}.

Maybe useful notes:

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$
$$Y_{21} = -\sqrt{\frac{15}{2\pi}} \sin\theta \cos\theta \; e^{i\phi}$$
$$Y_{33} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}} \sin^3\theta \; e^{3i\phi}$$
$$Y_{31} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}} \sin\theta(5\cos^2\theta - 1)e^{i\phi}$$
$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \; e^{i\phi}$$
$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \; e^{2i\phi}$$
$$Y_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\theta - 1)$$
$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \sin^2\theta \cos\theta \; e^{2i\phi}$$
$$Y_{30} = \frac{1}{2}\sqrt{\frac{7}{4\pi}} \sin\theta(5\cos^3\theta - 3\cos\theta)$$

$$Y_{lm}(\theta, \phi) \propto P_l^m(\cos\theta)e^{im\phi}$$

$$P_l^m(x) = (-1)^m(1 - x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

$$(l + 1)P_{l+1} - (2l + 1)xP_l + lP_{l-1} = 0$$

$$P_0(x) = 1; \quad P_1(x) = x$$

$$\bigtriangledown^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\ell(\ell + 1)}{r^2}$$

6. Calculate the expected amplitude of the tides (measured between high tide to low tide), in meters, caused by the differential gravitational pull from the Moon, under the assumption that the entire Earth were covered with an ocean of uniform thickness of 4 km. How does this compare to the amplitude of the tides due to the Sun?

7. A nuclear bomb exploded high above the earth creates an electromagnetic pulse. Model the blast wave of the bomb as a perfectly conducting plasma sheet travelling perpendicular to the earth's magnetic field with velocity v. What is the electric field generated? (Earth's magnetic field strength is 50×10^{-6} T.)

8. Complex index of refraction

A. Absorption in an optical medium can be modelled by giving the medium a complex index of refraction. Light travelling in vacuum ($\lambda = 500$ nm) is normally incident onto a semi-infinite slab of material with a complex index of refraction $n = 1.4 + 0.4i$. What fraction of the energy of the light is absorbed by the material?

B. What fraction of the same normally incident light would be absorbed by a 500 nm thick layer of this same material, if the material is in vacuum?

September 2011 Physics & Astronomy Qualifying Exam
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Day 2: September 2, 2011

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charge of an electron	e	4.8×10^{-10} esu
electron volt	1 eV	1.6×10^{-19} J
Larmor formula		$P_{rad} = (\ddot{\vec{p}})^2/6\pi\epsilon_0 c^3$
mass of an electron	m_e	0.511 MeV/c^2
mass of a proton	m_p	938 MeV/c^2
mass of the Sun	M_{sun}	2×10^{30} kg
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
Planck's constant, reduced	$\hbar$	1.1×10^{-34} J·s
radius of the Sun	R_{sun}	7×10^8 m
specific heat of water	C	4186 J/kg/°C
speed of light	c	3.0×10^8 m/s
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9. A spherical piece of ice ($T = 0°C$) with radius 50 mm is immersed in a much larger tank of water at $T = 10°C$. A piece of string extending from the center of the ice is attached to the bottom of the tank, which keeps the ice from floating to the top. After 5 minutes the tension on the string has decreased from 0.0524 N to 0.0463 N. How long until the tension on the string decreases to one-quarter of its original value?

10. An astronomer uses Doppler shifts to measure the radial velocities of a large number of stars in a spherical cluster of stars. The cluster is estimated to be 40 light-years in diameter. By looking at the Lyman α emission line, the RMS spread of the measured wavelengths is observed to be 0.1% of the mean. Estimate the total mass of the galactic cluster, using the virial theorem. (Reminder: the virial theorem says $\langle 2K + U \rangle = 0$, where K and U represent kinetic and potential energy. You may assume that all stars have the same mass, and that the number density of stars within the cluster is approximately constant as a function of radius.

11. Four identical dipole radio antennas emitting at wavelength λ are oriented in the vertical direction. They are arranged in a straight line along the ground with a spacing of $\lambda/2$ between antennas. Originally all four antennas are driven in phase, so that the maximum power is emitted at an angle of 90 degrees to the line of the antennas.

A. An observer stands far from the antennas at 90 degrees (maximum intensity), and starts walking in a circle. After he has moved 10 degrees, by what factor has the detected power dropped?

B. The observer continues to walk in a circle. At what angle will the power first reach zero?

C. The direction of the beam can be steered by inserting adjustable time delays between the signals to the antennas. If it is desired to steer the beam so that instead of being maximum at $90°$ it is maximum at $60°$ then what time delay should be inserted, if $\lambda = 3$ m?

12.

A. Starting from the nonrelativistic Larmor formula (in MKS units) for the total power radiated by an accelerated charge

$$P = \frac{1}{6\pi\epsilon_0}\frac{q^2}{c^3}\left|\frac{dv}{dt}\right|^2$$

write a relativistic generalization for this power which is a Lorentz scalar and reduces to the Larmor formula in the nonrelativistic limit.

B) Now consider a relativistic ($\gamma \gg 1$) charged particle in uniform circular motion at constant speed, and calculate the total power radiated in terms of the radius R of the orbit and the energy E of the particle.

C) For the case of a 10 GeV electron in a 100 m radius circular accelerator moving at constant speed, calculate the energy radiated per revolution in eV.

13. A particle of mass m and charge e is suspended on a string of length L above an infinite plane conductor. The distance of closest approach is a. Neglect the force of gravity.

A. Compute the frequency of this pendulum for small oscillations.

B. Compute the power radiated per unit solid angle, $dP/d\Omega$, as a function of angles for small oscillations of linear amplitude X_0.

C. Compute the total power radiated (take the case $\lambda \gg a$), in terms of e,X_0,a,m, and L.

14. A beam of atoms of mass m and energy E is passed through a small hole of diameter d in an opaque plate normal to the beam. The atoms are then detected by a second plate a distance L away. Under the most ideal circumstances, what is the lower bound for the diameter D of the spot that the beam forms on the detector plate? L and E are fixed, but the hole diameter d may be varied. Take the beam cross section to be always much wider than the hole. Notice that, as $d \to 0$, the uncertainty in momentum will lead to $D \to \infty$, while, as $d \to \infty$, clearly D will also become very large. You are asked to find the optimal hole size d for which D is the smallest.

15. In this problem we would like you to consider how long it takes (order of magnitude) for a star to relax if we perturb its temperature — or equivalently the time to transport heat from one point to another. We consider three possible cases.

a) If the process is radiative, photons carry the heat. Photons are easily absorbed, reradiated and scattered in a stellar interior and their mean free path is very small - in the Sun it is about 1 cm. If the radius of the Sun is 7×10^{10} cm, estimate the relaxation time to smooth out a temperature perturbation in the Sun.

b) If the process is convective, heat is carried by convective currents. In this case a buoyancy force accelerates a blob of material upwards and returns cooler material down towards the centre of the star. Assuming that all this happens at constant pressure, solve for the time it would take to smooth out a 1 degree fluctuation near the middle of the Sun (mass interior = 10^{33} grams, temperature = 10^7 K, giving the convective current a length of $1/10^{th}$ of the radius of the Sun.

c) Finally, if the centre of the star is degenerate, the energy is carried by the degenerate electrons. Assume that the degenerate core of a star is 1/100 of the radius of the Sun, and that the temperature is again 10^7 K, estimate how long it would take to transport the heat.

16. Life on earth would be difficult if not impossible if exposed to radiation of solar, charged particles. We are being protected from them by the magnetic field of the earth. They have a typical energy spectrum of $d\Phi/dE \propto E^{-3}$ particles/m^2/s/J. Estimate the minimal strength of this field following the anthropic principle (i.e. the field can't be weaker or else we wouldn't be here to observe the phenomenon).