Astronomy Comprehensive Exam, Spring 2023

Session 1

May 12, 2023

Note: if you are in the PhD in physics program, stop! This is the astronomy version of the exam. Please ask the proctor for the version appropriate for your program instead.

Do not write your name on your exam papers. Instead, write your student number on each page. This will allow us to grade the exams anonymously. We'll match your name with your student number after we finish grading.

This portion of the exam has 4 questions. Answer ***any three*** of the four. Do not submit answers to more than 3 questions—if you do, only the first 3 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 2 hours 15 minutes to complete 3 questions.

You are allowed to use two $8.5'' \times 11''$ formula sheets (both sides), and a handheld, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	-273°C
atomic mass unit	1 amu	1.661×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
distance from earth to sun	1 AU	1.5×10^{11} m
Laplacian in spherical coordinates	$\nabla^2\psi =$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$
mass of an electron	m_e	0.511 MeV/c^2
mass of hydrogen atom	m_H	1.674×10^{-27} kg
mass of a neutron	m_n	1.675×10^{-27} kg
mass of a proton	m_p	1.673×10^{-27} kg
mass of the sun	M_{sun}	2×10^{30} kg
molecular weight of H_2O		18
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
nuclear magneton	μ_N	5×10^{-27} J/T
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the Earth	R_{earth}	6.4×10^6 m
radius of a neutron	$R_{neutron}$	3×10^{-16} m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Stirling's approximation	$N!$	$e^{-N}N^N\sqrt{2\pi N}$

1. A sphere of mass m and radius r hangs from a massless string of length L. A very light but steady breeze blows in the horizontal direction with speed v_0. The ball experiences a drag force that depends on its velocity relative to the air, the mass density of the air (ρ), the cross-sectional area of the ball, and a dimensionless coefficient of drag C_D.

   (a) Calculate the angle θ_0 relative to vertical at which this pendulum will hang motionless.

   (b) Now assume that the sphere is slightly displaced from θ_0 to $\theta_0 + \epsilon$, and then released. Calculate the damped motion $\theta(t)$ in the limit of small ϵ.

2. Consider a 2D quantum mechanical simple harmonic oscillator with potential given by $V(r) = \frac{1}{2}m\omega^2 r^2$. The eigenfunctions of this system, expressed in polar coordinates, can be shown to be of the form $\Psi(r,\theta) = R_{\ell k}(r)e^{i\ell\theta}$, where ℓ is any integer, k is any non-negative integer ($k = 0, 1, 2, 3, ..$), and $R_{\ell k}(r)$ is the radial eigenfunction. The energy eigenvalues are then given by $E = A(Bk + |\ell| + C)$. Determine the values of the coefficients A, B, and C. (Hint: this problem can be solved without solving for the radial eigenfunctions!)

3. Consider a system of two identical electrons which are confined to move on a circle of radius R. The electrons are charged and R is small enough that their electromagnetic repulsion is important.

   The dynamics of the electrons is described by the Hamiltonian

$$H = \frac{|\vec{p}_1|^2}{2m} + \frac{|\vec{p}_2|^2}{2m} + \frac{e^2}{4\pi\epsilon_0|\vec{x}_1 - \vec{x}_2|}$$

   where we have assumed that spin-dependent interactions are negligible. Once constrained to the circle, the coordinates are $\vec{x}_i(t) = (R\sin\phi_i(t), R\cos\phi_i(t), 0)$.

   In a semi-classical approach to the problem of finding the low energy states of this system we would start by finding a classical solution for the positions of the electrons and study the quantum mechanics of small deviations from that classical solution. Let us assume that this procedure will give an accurate result.

   This system (known as the "Wigner molecule") can either have spin 1 or spin 0, depending on how the spins of the electrons are aligned. It will have quantized rotational and vibrational excitations.

   Find the rotational and vibrational energy spectra of this molecule as a function of $R, \hbar, e, m$. How do they depend on the spin of the molecule?

   Hint: consider reparametrizing the angular positions in terms of $\xi_A = \frac{1}{2}(\phi_1 + \phi_2)$ *and* $\xi_B = \frac{1}{2}(\phi_1 - \phi_2)$.

4. Two stars with equal masses M orbit each other in a circular orbit. The initial distance between the stars at time t_0 is R_0. The system will emit gravitational radiation with a power of

$$L \approx A\frac{M^3}{R^5}$$

   where A is a proportionality constant. Calculate the time dependence of the distance between the stars and the time derivative of the orbital frequency as a function of time. You may assume the masses don't change over time.

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Session 2

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Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	$-273.16°$C
astronomical unit	au	$149,597,870.7$ km
atomic mass unit	1 amu	1.66×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
luminosity of the sun	$L_\odot$	3.8×10^{26} W
mass of an electron	m_e	0.511 MeV/c^2
mass of hydrogen atom	m_H	1.674×10^{-27} kg
mass of a neutron	m_n	1.675×10^{-27} kg
mass of a proton	m_p	1.673×10^{-27} kg
mass of the earth	$M_\oplus$	5.97219×10^{24} kg
mass of the sun	$M_\odot$	2×10^{30} kg
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
parsec	pc	3.086×10^{16} m
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the earth	$R_\oplus$	$6,371.0$ km
radius of Jupiter	$R_{♃}$	$69,911$ km
radius of the sun	$R_\odot$	$696,342$ km
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Thomson cross section	σ_T	6.65×10^{-29} m^2

1. Consider a spherical star of mass M and radius R.

   (a) Assuming that the density is constant throughout the star, determine the internal pressure as a function of distance r from the centre. Express your answer in terms of M and R. What value does your result give for the central pressure in the Sun?

   (b) Assuming that the star is non-degenerate and that the composition at the centre is 50% helium and 50% hydrogen (mass ratio), estimate the central temperature. What temperature does this give you for the centre of the Sun?

2. Assume that the Earth is a solid sphere of radius R and mass M, and has an internal density $\rho(r) = \rho_0(1 - r^2/R^2)$, where ρ_0 is a constant. Suppose that you bore a hole through to the centre of the Earth and drop an egg into it. Neglecting air resistance, what is the speed of the egg when it reaches the centre of the Earth? Express your results in terms of M and R, and then calculate the numerical value using the Earth's mass and radius.

3. Consider an accretion disk of outer radius R surrounding a black hole of mass M. Gas accretes onto the outer edge of the disk at a constant rate $\dot{m}$ (mass per unit time). The disk rotates with Keplerian velocity and slowly transfers momentum outwards via viscosity. This heats the disk which then radiates like a black body. At the inner edge of the disk, at 3 Schwarzschild radii, gas falls into the black hole.

   (a) Assuming that all the gravitational energy of the disk is converted to heat and radiated, estimate the luminosity of the disk. In the limit $R \to \infty$, what fraction of the rest mass energy of the gas is converted to radiation? You can assume that the mass of the disk is much less than that of the black hole.

   (b) Estimate the temperature of the disk, as a function of radius r. (Hint: determine the luminosity radiated between r and $r + dr$ and from that find the flux emitted by that annulus)

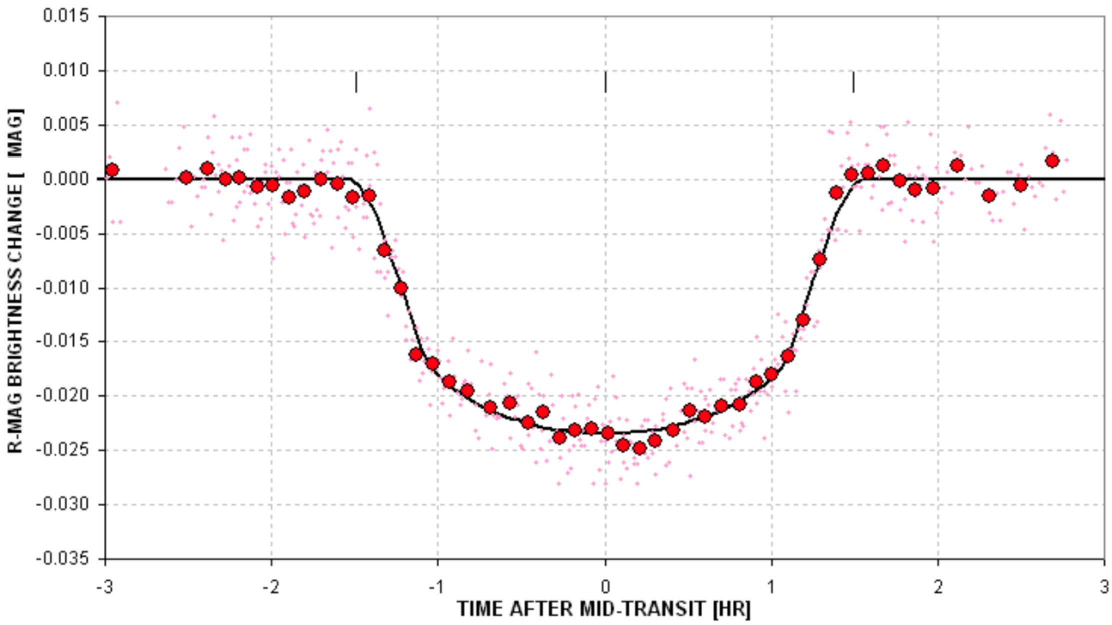
4. The figure on the next page shows the light curve of a sun-like star showing a planetary transit. Assuming that the transit is central, and that the planet is in a circular orbit, estimate

   (a) the radius of the planet, in units of jupiter radii,

   (b) the period in days and radius in au of the planet's orbit,

   (c) the average surface temperature of the planet.

   You may ignore limb darkening.


0.015
0.010
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-0.035
R-MAG BRIGHTNESS CHANGE [MAG]
-3
-2
-1
0
1
2
3
TIME AFTER MID-TRANSIT [HR]