Astronomy Comprehensive Exam, Fall 2021
Session 1

May 13, 2022

Note: if you are in the PhD in physics program, stop! This is the astronomy version of the exam. Please download the physics version instead.

Do not write your name on your exam papers. Instead, write your student number on each page. This will allow us to grade the exams anonymously. We'll match your name with your student number after we finish grading.

This portion of the exam has 4 questions. Answer ***any three*** of the four. Do not submit answers to more than 3 questions—if you do, only the first 3 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 2.25 hours to complete 3 questions.

You are allowed to use two $8.5'' \times 11''$ formula sheets (each written on both sides), and a handheld, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	-273.16°C
atomic mass unit	1 amu	1.66×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
distance from earth to sun	au	1.5×10^{11} m
luminosity of the sun	$L_\odot$	3.8×10^{26} W
mass of an electron	m_e	0.511 MeV/c^2
mass of hydrogen atom	m_H	1.674×10^{-27} kg
mass of a neutron	m_n	1.675×10^{-27} kg
mass of a proton	m_p	1.673×10^{-27} kg
mass of the sun	$M_\odot$	2×10^{30} kg
molecular weight of H_2O		18
molecular weight of N_2		28
molecular weight of O_2		32
weight of Helium atom He		4
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
parsec	pc	3.086×10^{16} m
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the earth	$R_\oplus$	6.4×10^6 m
radius of the sun	$R_\odot$	7×10^8 m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Thomson cross section	σ_T	6.65×10^{-29} m^2

1. A particle traveling along the positive x axis of reference frame S with speed $0.5c$ decays into two identical particles, $a \to b + b$, both of which continue to travel on the x axis. Given that $m_a = 2.5 m_b$, find the speed of either b particle in the rest frame of particle a. Then, by making the necessary transformation on the result, find the velocities of the two b particles in the original frame S.

2. Two identical elastic balls 1 and 2 are initially at two ends of a one-dimensional frictionless track that is 1 m long, and move toward each other with speeds v_1 and v_2 respectively. The balls can collide with each other and the two ends of the track elastically. Under which conditions can both balls reach their initial positions at the same time with their initial speeds? (Hint: draw a plot showing the world lines (position vs. time) of the two balls.)

3. A basketball has a diameter of 24 cm when inflated, and the wall of the basketball is 3 mm thick. When inflated to a gauge pressure of 55 kPa and placed on a scale at sea level, the scale reading is 625 g.

   (a) What reading would the scale give if all of the air were removed from this basketball? (You may assume that the ball collapses to its minimal possible volume.)

   (b) The inflated ball is set spinning at 3 rotations per second. What is the quantum limit on how accurately the angle of the spin axis can be aligned to vertical?

4. The solar constant expresses the intensity of radiation from the sun at the Earth's orbit, and is approximately 1400 Wm^{-2}. A "sail" made from a reflective mylar film 1 μm in thickness (the density of mylar is 1400 kg m^{-3}) is deployed in a spacecraft in a distant orbit around the earth, rotating such that it remains perpendicular to (normal to) the light from the sun at all times. The sail is initially orbiting the sun in the Earth's orbit, but far enough from the Earth that the effects of Earth's gravity may be neglected. What are the radial and tangential components of the sail's velocity when it crosses the orbit of Mars? (The radius of the Earth's orbit is 1.5×10^8 km, while Mars' orbit is at 2.3×10^8 km.)

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charge of an electron	e	1.6×10^{-19} C
distance from earth to sun	au	1.5×10^{11} m
Laplacian in spherical coordinates	$\nabla^2 f =$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
luminosity of the sun	$L_\odot$	3.8×10^{26} W
mass of an electron	m_e	0.511 MeV/c^2
mass of hydrogen atom	m_H	1.674×10^{-27} kg
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permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the earth	$R_\oplus$	6.4×10^6 m
radius of the sun	$R_\odot$	7×10^8 m
speed of the sun (galactocentric)	$v_\odot$	220 km s^{-1}
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Thomson cross section	σ_T	6.65×10^{-29} m^2

1. For a homogeneous and isotropic universe described by the Roberston-Walker metric

$$d\tau^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]$$

and filled with a fluid of mass-energy density ρ and pressure P, Einstein's equations reduce to two Friedmann equations,

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P), \end{aligned}$$

where the dot indicates a derivative with respect to coordinate time t. (We are using units in which the speed of light $c = 1$.) Show that the second equation can be derived from the first using the first law of thermodynamics (conservation of energy).

2. X-ray photons are produced in a spherical cloud of radius R at at a constant rate Γ (photons per unit volume per unit time), where r is the distance from the centre of the cloud and r_0 is a constant. The cloud is a distance d away. Assume that the cloud is optically thin. A detector at Earth has an angular acceptance beam of half-angle $\Delta\theta$ and an effective area A.

   (a) If the cloud is fully resolved by the detector, what is the observed intensity of the radiation as a function of position? What is the maximum intensity?

   (b) If the cloud is fully unresolved by the detector, find a lower limit to the average intensity. Explain why this is a lower limit.

3. (a) Show that the radial pressure profile for a uniform density, self-gravitating hydrostatic sphere is given by

$$P(r) = \frac{2\pi}{3}\rho^2 G(R^2 - r^2)$$

   for material density ρ and object radius R.

   (b) If the sphere is an ideal gas, find the temperature profile $T(r)$.

   (c) Estimate the central temperature if the sphere is pure hydrogen and has a mass of $1M_\odot$ and a radius of $1R_\odot$.

4. Globular clusters evolve over the age of the universe because they several relaxation times old. Assume here that there is a single stellar mass $m = 0.75M_\odot$ for all stars in the globular cluster, with a radial scale of 10 pc and total population $N_{tot} \sim 3 \times 10^5$.

   (a) Estimate the rms speed of stars in the cluster (in km/s), aiming for 50% accuracy.

   Late in the evolution, a subset of the stars form a contracting core (take the core to be of 1 pc radius and contain 10^4 white dwarves of radii about 1% $R_\odot$). The core collapse is stopped by a formed binary providing an energy source; later stellar encounters between the formed binary and a passing single star tend to give the passing star more kinetic energy which makes the binary's orbital separation drop (making the binary's energy more negative). By the epoch in cluster evolution that this happens, a likely binary to form is made of two white dwarves confined in the contracting core.

(b) Making sure not to mistakingly use specific quantities (that is, not energies per unit mass), show that the *total* orbital 2-body energy of a *single* white dwarf binary of sufficiently-small binary semimajor axis a can equal the gravitational *potential energy* of the entire collapsing core $W \sim GM^2/(2R)$, and compute that critical value of a for which equality occurs. Assume the binary star mutual orbits to be circular around their centre of mass, with total separation a. Explain quantitatively if is there another order of magnitude of energy still available in continuing shrinkage of the binary's mutual semimajor axis.