Ph. D. Comprehensive Exam 2005
Department of Physics and Astronomy
University of British Columbia

Part $\mathcal{I}$

Date and time: 09/15/2005 01:00 PM-05:00 PM
Room: SCRF 100

This exam is closed book. No notes, no books, no calculators no computers.
Please attempt all problems. All problems have equal weight. All sub-problems of each problem have equal contribution to the weight of the problem. Please label each problem clearly and write clearly in the answer book that is provided. Show your work. Explain the steps.

1. Consider N distinguishable particles each of which occupies one of two levels $i = 1, 2$. $\epsilon_i (i = 1, 2)$ is the energy of level i. The total energy for a given distribution (N_1, N_2) is

$$E(N_1, N_2) = N_1\epsilon_1 + N_2\epsilon_2 \tag{1}$$

where N_i, $i = 1, 2$ are the numbers of particles at level i, $N_1 + N_2 = N$.

(a) Calculate $\Omega(N_1, N_2)$, the number of microscopic states with N_i particles at level $i = 1, 2$.

(b) Using the Boltzman statistics, calculate Z, the partition function for N particles. Your results should be expressed as a function of $\epsilon_{1,2}$, kT and N. (There are a few different ways to evaluate this quantity). Z is defined as

$$Z = \sum_s \exp(-\beta E_s), \beta = \frac{1}{kT} \tag{2}$$

and the sum is over all microstates for N particles and E_s is the energy of state s.

(c) Calculate the average and fluctuations of N_1: $< N_1 >$ and $< (\delta N_1)^2 >=< (N_1- < N_1 >)^2 >$. Your results should be expressed as a function of $\epsilon_{1,2}$, kT and N.

2. Two homogeneous discs of masses m_1 and m_2, both of radius R, have centers connected by a spring so that they can roll without slipping. At the initial moment the centers are at $x_1(0) = 0$, $x_2(0) = 2L_0$ and have initial speeds $\dot{x}_1(0) = -v_0$, $\dot{x}_2(0) = 2v_0$. The initial conditions are such that the wheels never collide. Find their positions at all later times. The spring has an unstretched length L_0, and a spring constant

$$k = \frac{9v_0^2 m_1 m_2}{2L_0^2(m_1 + m_2)}$$

3. Relativistic motion under a constant force:

   Consider a relativistic particle of mass m and charge q in a constant electric field $\vec{E}$.

   (a) Write the Lagrangian for this system of a single charged relativistic particle moving in a constant electric field.

   (b) Obtain and solve the equation of motion. Assume that at time $x^0 = 0$, the position is $\vec{x}(0) = 0$ and the velocity vanishes, $\frac{d\vec{x}(0)}{dx^0} = 0$.

4. Particles of mass m are incident on the spherically symmetric Yukawa potential

$$V(r) = V_0 \frac{e^{-\mu r}}{r}$$

   where V_0 and μ are constants.

   (a) To lowest order perturbation theory, show that the differential cross section for the scattering vector k is given by

$$\frac{d\sigma}{d\Omega} = \left[\frac{2mV_0}{\hbar^2(k^2 + \mu^2)}\right]^2$$

   (b) Use this result to derive the Rutherford formula for the scattering of α-particles of energy E, incident on a nucleus with the atomic number Z, and being scattered at an angle θ to the incident direction.

5. A harmonic oscillator is in the state $|\Psi(0)> = \frac{1}{\sqrt{2}}(|0> + |2> + |4>)$ at $t = 0$.

   (a) Express $|\psi(t)>$ for arbitrary time.

   (b) Compute $< x >$ and $< x^2 >$ for arbitrary t.

   Hint: recall that $a = \sqrt{\frac{m\omega}{2\hbar}}(x + ip/m\omega)$.

6. Consider the degenerate electron gas in a metal as a mixture of two gases of spin-up and spin-down electrons, respectively. when a small magnetic field B is applies, a few of the electrons reverse their spins so as to maintain equality of the chemical potential in the two mixed gases. For $T = 0$, find the magnetic susceptibility of the metal $(\partial M/\partial B)_{N,V}$, where M is the magnetization (magnetic moment per unit volume), N is the total number of electrons, and V is the volume. The magnetic moment of the electron is μ_B.

Part *II*

Date and time: 09/16/2005 01:00 PM 05:00 PM
Room: FNSC 60

This exam is closed book. No notes, no books, no calculators no computers. Please attempt all problems. All problems have equal weight. All sub-problems of each problem have equal contribution to the weight of the problem. Please label each problem clearly and write clearly in the answer book that is provided. Show your work. Explain the steps.

1. Dipole radiation: Consider purely harmonic sources of electromagnetic radiation:

$$\rho(\vec{r},t) = \rho(\vec{r})e^{-i\omega t} + \rho^*(\vec{r})e^{i\omega t}$$

$$\vec{j}(\vec{r},t) = \vec{j}(\vec{r})e^{-i\omega t} + \vec{j}^*(\vec{r})e^{i\omega t}$$

A similar decomposition of the $\vec{r}$- and t-dependence will also hold for the potentials and the field strengths. These sources are contained in a region of overall dimension a, centered at the origin.

(a) Suppose that you go very far away from this localized current distribution in the direction given by the unit vector $\hat{r}$. Using the Lorentz gauge, show that the field strengths in this "far field", or radiation zone are given by

$$\vec{B}(\vec{r}) = i\vec{k} \times \vec{A}(\vec{r})$$

$$\vec{E}(\vec{r}) = -\frac{i}{k}\vec{k} \times \left(\vec{k} \times \vec{A}(\vec{r})\right)$$

Here $|\vec{r}| >> a$, and the vector $\vec{k}$ has direction $\hat{r}$ and magnitude $|\vec{k}| = \omega/c$.

(b) Show that in this far zone the vector potential is given by

$$\vec{A}(\vec{r},t) = \frac{e^{-i\omega(t-|\vec{r}|/c)}}{c|\vec{r}|}\vec{j}(\vec{k}) + \text{ complex conjugate}$$

where $\vec{j}(\vec{k})$ is the Fourier transform of the current distribution

$$\vec{j}(\vec{k}) = \int d\vec{r}'\vec{j}(\vec{r}')e^{-i\vec{k}\cdot\vec{r}'}$$

(c) From (a) and (b) above, obtain Poynting's vector in the far zone,

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} = \frac{c}{4\pi}\vec{E}^2(\vec{r},t)\hat{r}$$

Putting $\vec{E} = \vec{e}_1 E_1 + \vec{e}_2 E_2$, where $\vec{e}_1$ and $\vec{e}_2$ are certain polarization vectors, derive the time-averaged power radiated in the direction $\hat{r}$ into the solid angle $d\Omega$,

$$\frac{dP}{d\Omega} = \frac{\omega^2}{2\pi c^3}\sum_\lambda \left|\vec{e}^*_\lambda \cdot \vec{j}(\vec{k})\right|^2$$

Hint: The retarded potentials are given by

$$\Phi(\vec{r},t) = \int \frac{\rho(\vec{r}',t-|\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|} \quad \vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{\vec{J}(\vec{r}',t-|\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}$$

2. The tension τ in a stretched wire of length L and radius r is increased reversibly from τ_0 to τ_1 at constant temperature $T = T_0$. Assume a linear expansivity, $\alpha = \frac{1}{L}\left(\frac{\partial L}{\partial T}\right)_\tau$ and the isothermal Young's modulus $Y \equiv \frac{L}{\pi r^2}\left(\frac{\partial \tau}{\partial L}\right)_T$ are constant.

   (a) Show

   $$TdS = C_T dT + T\left(\frac{\partial L}{\partial T}\right)_\tau d\tau$$

   where S is the entropy and $C_\tau \equiv T\left(\frac{\partial S}{\partial T}\right)_\tau$ is the heat capacity at constant tension.

   (b) What is the change in internal energy? (Hint: A stretched wire can be described thermodynamically using τ, L and T as state variables. this description may be developed in complete analogy to the description of a pure substance using T, the pressure P and the volume V where $P \rightarrow -\tau$ and $V \rightarrow L$. For example, the first Law is $dE = TdS + \tau dL$ and the Gibbs free energy is $G = E - \tau L - TS$.)

3. The fractional energy fluctuation $< \Delta E >_{rms} \;/ < E >$ can be determined within two common models:

   (a) For an Einstein solid at temperature T with Einstein frequency ω_E, find $< \Delta E >_{rms}$ $/ < E >$. At what temperature are fluctuations largest?

   (b) For a Debye solid with Debye frequency ω_D, derive expressions for the low-temperature and high-temperature limits of the fractional energy fluctuation (integrals need not be evaluated but should be reduced to dimensionless variables).

4. Show that in the LAB system the relativistic expression for the kinetic energy of a particle scattered through an angle ψ by a target of equal mass is

   $$\frac{T_1}{T_0} = \frac{2\cos^2\psi}{(\gamma_1+1)-(\gamma_1-1)\cos^2\psi}$$

   The subscripts 0 and 1 correspond to the initial and final states of the incident particle, γ is the Lorentz factor. The expression reduces to the non-relativistic result when $\gamma_1 \rightarrow 1$. Sketch $T_1(\psi)$ for neutron-proton scattering for incident neutron energies of 100MeV, 1GeV, and 10GeV.

5. There exist oscillatory systems which are not closed but in which the external action amounts to a time variation of parameters. For instance, consider the equation of motion for a one-dimensional system

   $$\frac{d^2x}{dt^2} + \omega^2(t)x = 0$$

   where $\omega(t)$ is a function of time. Assuming

   $$\omega^2(t) = \omega_0^2(1 + h\cos\gamma t)$$

   with $h << 1$, find

(a) the solution of the equation of motion to the first non-trivial order in h.

(b) the range of frequencies γ where parametric resonance occurs.

(c) for what frequency γ is the parametric resonance the strongest?

6. A particle of mass m is restricted to move in the vertical direction in the earth's gravitational field. Assume that the surface of the earth reflects this particle elastically (like a steel ball falling on a surface of glass) and go through the following steps to quantize the energy of the motion using the momentum representation instead of position representation:

   (a) Set up the Hamiltonian, assuming that the gravitational potential is zero on the surface of the earth.

   (b) Set up the Schrödinger equation in the momentum representation.

   (c) Solve this Schrödinger equation for the particle in the momentum representation. Do not use any boundary conditions yet, but use the abbreviations

   $$\frac{2m^2 g}{\hbar^2} = \frac{1}{\ell^3} \quad \frac{2mE}{\hbar^2} = \frac{\lambda}{\ell^2}$$

   where it is convenient.

   (d) From the momentum space eigenfunctions, obtain the position space eigenfunctions $\psi(x)$.

   (e) The integral representation of the Airy function is

   $$Ai(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(iu^3/3 + iuz) du$$

   Express your $\psi(x)$ in terms of this function – no need to normalize.

   (f) Realizing that the elastic reflection means that $\psi(x = 0) = 0$, fund the transcendental equation that quantizes the energy. How would you use it to obtain the energy levels?