Autumn 2019 Physics & Astronomy Qualifying Exam
for Advancement to Candidacy
Day 1: August 29, 2019

Do not write your name on the exam. Instead, write your student number on each exam booklet. This will allow us to grade the exams anonymously. We'll match your name with your student number after we finish grading. If you use extra exam booklets, write your student number on the extra exam books as well. Write all answers in the blank exam booklet(s), not on this printout!

Today's portion of the exam has 8 questions. Answer *any five* of the eight. Do not submit answers to more than 5 questions—if you do, only the first 5 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 4 hours to complete 5 questions.

You are allowed to use one 8.5″ × 11″ formula sheet (both sides), and a handheld, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	-273°C
atomic mass unit	1 amu	1.66×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
distance from earth to sun	1 AU	1.5×10^{11} m
Laplacian in spherical coordinates	$\nabla^2 f =$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
mass of an electron	m_e	0.511 MeV/c^2
mass of hydrogen atom	m_H	1.674×10^{-27} kg
mass of a neutron	m_n	1.675×10^{-27} kg
mass of a proton	m_p	1.673×10^{-27} kg
mass of the sun	M_{sun}	2×10^{30} kg
molecular weight of H_2O		18
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
nuclear magneton	μ_N	5×10^{-27} J/T
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the Earth	R_{earth}	6.4×10^6 m
radius of a neutron	$R_{neutron}$	3×10^{-16} m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Stirling's approximation	$N!$	$e^{-N}N^N\sqrt{2\pi N}$

1. Neutron stars form when a solar mass is compressed to a radius $R \sim 10$ km. Estimate the maximum spin rate of a neutron star. Express your answer in revolutions per second. Estimate as well the linear velocity at the equator of the star.

2. This problem explores adding the spin angular momenta of three distinguishable spin-1/2 particles. Consider the following five wavefunctions:

$$\frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle)$$
$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle)$$
$$\frac{1}{\sqrt{3}}(|\uparrow\uparrow\downarrow\rangle + e^{2i\pi/3}|\uparrow\downarrow\uparrow\rangle + e^{-2i\pi/3}|\downarrow\uparrow\uparrow\rangle)$$
$$|\downarrow\uparrow\uparrow\rangle$$
$$|\uparrow\uparrow\uparrow\rangle$$

A. Which of these wavefunctions are eigenfunctions of the total spin angular momentum operator s^2_{total}? Which are not?

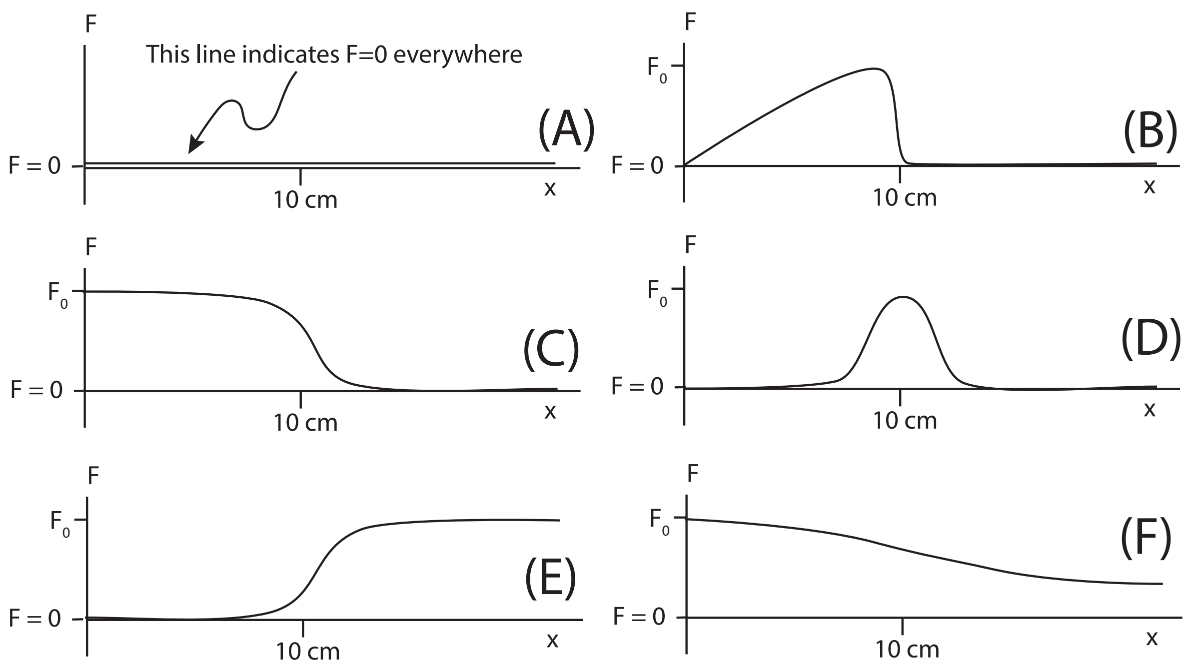
B. For those which are eigenfunctions of the total spin, state the eigenvalue of s^2_{total} for each.

C. For three spin-1/2 particles there are of course $2^3 = 8$ possible states. Write down eight mutually orthogonal states that are each eigenstates of the total angular momentum. (Some of these are given above, but not all.)

A parallel plate capacitor is formed from two thin square copper plates, 20cm on a side, each extending from +10cm to -10cm on the x and y axes. The plates are located at z=+1cm and z= -1cm respectively. A +1C charge is added to the top plate, which is electrically isolated, while the bottom plate is grounded.

3. A 1mm-diameter sphere of charge, 1μC, is brought in towards the origin along the x-axis (y=0, z=0). For each of the x,y, and z components of the force on the sphere, F_x, F_y, and F_z, choose the sketch that best represents the x-dependence of the force, and provide an estimate for the vertical axis tick mark shown, F_0. Note that F_0 may be negative. (No need to be exact, but do as well as you can.)

Then, two 1mm-diameter spheres of charge, one with +1μC and the other with -1μC, are fixed together in a vertical dumbell orientation, with the positively-charged sphere centred at z=+1mm (y=0) and the other at z=-1mm (y=0). The vertical dumbell is brought in towards the origin along the x-axis, as above. As above, choose the sketch that best represents the x-dependence of the force on the dumbell and estimate the tick mark value.



F_x for single sphere: graph ________ ; F_0 ~ ________ N (Newtons)

F_y for single sphere: graph ________ ; F_0 ~ ________ N

F_z for single sphere: graph ________ ; F_0 ~ ________ N

F_x for double sphere: graph ________ ; F_0 ~ ________ N

F_y for double sphere: graph ________ ; F_0 ~ ________ N

F_z for double sphere: graph ________ ; F_0 ~ ________ N

Reminder: don't write your answers on this page, but in your exam booklet

4. Typical dust particles in the Solar System have mass density ρ. They experience both gravitational and radiation pressure effects of the Sun (mass $M = 2 \times 10^{30}$ kg, emitted light power $P = 4 \times 10^{26}$ W).

A. For macroscopically sized dust particles (those having sizes large compared to the wavelength of light) are the smaller or the larger particles more likely to be ejected from the Solar System?

B. Estimate (symbolically) the critical radius R of a particle that would not be ejected from the Solar System.

C. Provide a numerical estimate to part B.

D. Qualitatively speaking what happens in the limit that the dust particles are small compared to the wavelength of light?

5. What is the critical angle for total external reflection for photons of wavelength λ and frequency $\omega = 2\pi c/\lambda$ in vacuum, falling on a metal plate with electron density ρ? Assume that the photons are above the plasma frequency in the metal, so that the electrons can be approximated as free (i.e. not interacting with each other).

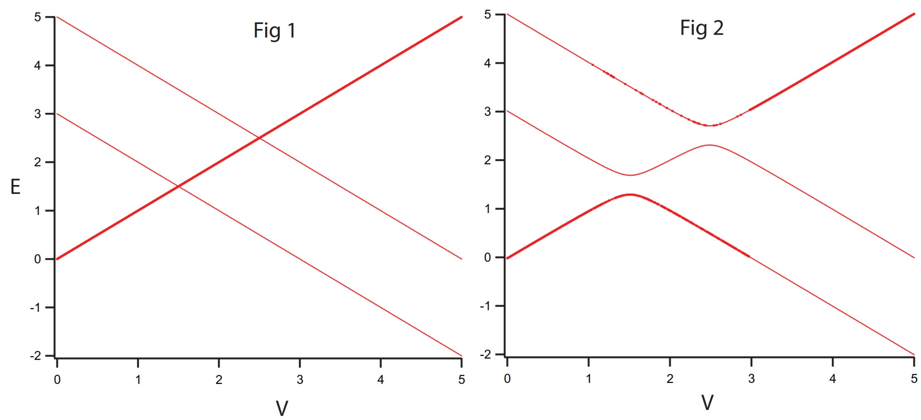
6.

A. Consider a 1-dimensional delta function potential well $V(x) = -aV_0\delta(x)$, where a and V_0 are positive constants. A point particle of mass m is bound in this potential. Show that there is only one bound state in this potential, and find its binding energy and the wavefunction of the bound state.

B. Now consider two symmetric delta function potential wells, $V(x) = -aV_0\left[\delta(x+a) + \delta(x-a)\right]$. Employing only a symmetry argument without solving the Schroedinger equations for this potential, guess the ground state wavefunction and first excited state wavefunction from the wavefunction obtains in Part A. It is not required to normalize wavefunctions in this problem.

C. Let $\lambda \equiv \frac{2mV_0}{\hbar^2}a^2$. Assuming that $\lambda \gg 1$, find the energy of the ground state in Part B up to the correction term to the answer you obtained in Part A.

7.



The Hamiltonian H of a three-state system can be expressed as a matrix as:

$$H(V) = \begin{pmatrix} V & 0 & 0 \\ 0 & 3-V & 0 \\ 0 & 0 & 5-V \end{pmatrix}$$

where energies are expressed in units of 10^{-25} J, using a basis where

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

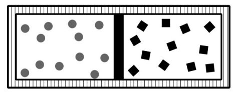
The eigenstates of this Hamiltonian have energies, as a function of V, as shown in Fig. 1.

Then, a perturbation h is added to H, such that the eigenstates of the new Hamiltonian $H + h$ have energies as shown in Fig. 2.

A. Write down a possible matrix representing h.

B. Suppose V is a parameter that can be controlled in the laboratory over the range shown in the graphs. Describe how you could ramp V smoothly and monotonically from 0 to 5 to (approximately) transform an initial state with energy of 0 into a final state with an energy of 0, assuming the system evolves under $H + h$. Make sure to specify any quantitative characteristics that your ramp must satisfy.

8. Consider a thermally insulated box with volume $2V_0$ containing two distinct monatomic ideal gases, separated by an impermeable barrier, as illustrated below.



On the left of the partition are N_0 atoms of a circular atom type, stored in volume V_0 at temperature T_0. On the right side of the partition are N_0 atoms of a square atom type, which occupy volume V_0 and are in thermal equilibrium with the circular atoms on the left side of the barrier.

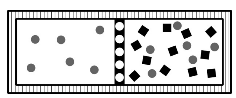
For this problem you may need the following equations of state dening the behavior of a mixture of two monatomic ideal gases:

$$U = \frac{3}{2} N k_B T$$

$$p = \frac{N k_B T}{V}$$

where N is the total number of atoms (squares plus circles) in a given volume.

A. The barrier between the two sides of the box is now made permeable to the circular atoms only, while remaining impermeable to the square atoms on the right side.

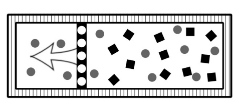


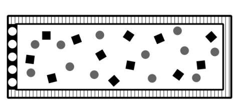
When the box has reached equilibrium, what is the pressure in each side of the box?

B. During this process, did the entropy of the system increase, decrease or remain the same? Explain your answer.

C. During this process, did the temperature of the system increase, decrease or remain the same? Explain your answer.

D. Now we will slowly move the permeable partition to the left side of the box, until it reaches the left-hand wall, at which point there will be only one enclosure with volume equal to $2V_0$.





Consider the change in entropy of the system (enclosed by the box) due to moving the permeable membrane to its edge. Is this change positive, negative or zero? Explain your answer.

E. What is the final temperature of the system?

Autumn 2019 Physics Qualifying Exam
for Advancement to Candidacy
Day 2: August 30, 2019

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radius of a proton	R_p	$1\times^{-15}$ m
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9. Lithium niobate has an index of refraction that changes linearly with an applied electric field E according to

$$n(E) = n_0 - \frac{1}{2} r n_0^3 E$$

Here $n_0 = 2.29$ is the index at zero field, and the coefficient $r = 3 \times 10^{-11}$ V/m.

Suppose a 1 cm × 1 cm × 1 cm cube of lithium niobate is placed between the two plates of a parallel plate capacitor. A sinusoidally varying voltage of amplitude 100 V and frequency 1 MHz is applied across the capacitor.

Laser light with wavelength $\lambda = 1000$ nm passes through the crystal, perpendicular to the applied field. As a result of the time-varying index of refraction, the beam acquires sidebands at different frequencies. If the electric field amplitude of the initial beam is represented by $Ae^{i\omega t}$, calculate the amplitudes and frequencies of the two most significant sidebands.

10. Our friend the Sun:

A. Estimate the mass of the Sun from your knowledge of its distance and any other common facts.

B. Estimate the radius of the Sun from everyday observations.

C. Estimate the mean interior pressure of the Sun. (You may assume here that the Sun has constant density.)

D. Estimate the mean interior temperature of the Sun.

E. Estimate the mean energy of a proton in the Sun.

F. Estimate the proton-proton Coulomb barrier that must be exceeded if two protons are to initiate a fusion reaction.

G. Considering the answers to (e) and (f), how is fusion able to proceed in the Sun?

11. A wire with mass m and length L is held under tension F, and clamped at either end. The wire can vibrate in the transverse directions.

A. Calculate the frequencies of the eigenmodes of vibration.

B. Calculate the total energy of the n^{th} vibrational mode as a function of its amplitude A_n.

C. Assuming that the wire is in thermal equilibrium, calculate the RMS amplitude of the n^{th} mode ($\sqrt{\langle A_n^2 \rangle}$). (You may ignore quantum effects, and treat this classically.)

12. Two identical bosonic atoms are subject to an external harmonic confining potential of the form $\frac{1}{2}M\Omega^2X^2$, where the $M = 2m$ is the total mass of two atoms and X is the position of their centre of mass. Assume they further interact with a simple harmonic interaction $\frac{1}{2}m\omega^2(\Delta x)^2$, where Δx is the distance between the atoms. Find out the excitation spectrum of these two particles in the external potential. (To make this problem easier, solve it in one spatial dimension.)

13. Water molecules in ice (molecular weight 18) are held in position in the ice's crystalline structure, but can oscillate about their nominal positions due to their thermal energy. As heat is added, the amplitude of the molecules' oscillation in position increases. In the simple but standard (Lindemann type) model of melting, melting occurs at the point at which the amplitude of these fluctuations is of the order of the intermolecular distance. Use this to do an order of magnitude estimate of the latent heat of the ice-water transition. (Hint: the bulk of the latent heat goes into breaking the crystal bonds so that, in water, molecule positions are disordered, unlike in ice.

14. A Chinook is a phenomenon in the Rocky Mountains occurring when strong winds blow from the west across the eastern slopes of the mountain range and rapidly descend into cities located at much lower elevation. Although the mountains are cold, the cities can warm quickly within a few minutes after the Chinook wind arrives. Sometimes the temperature can rise up by 20°C. Assume the air pressure at the top of mountain (4000 m elevation) is 0.6 atm and that the temperature is -15°C. The temperature in the city of elevation 1500 m is 0°C and the pressure is 0.8 atm. What will be the climb of temperature when the Chinook arrives? (Hint: you may treat the air as a diatomic ideal gas, and you can ignore energy transfer into bulk motion of the air.)

15. A spaceship with rest-mass m is moving with speed $v = 0.6c$ with respect to an inertial 'lab' frame. No external forces act on the spaceship. In order to decelerate, at time $t = 0$ the spaceship switches on a laser beam which is aimed in its direction of motion. The laser has power P_0 and it has emission frequency ν_0, both measured in the rest frame of the spaceship. What is the laser emission frequency ν measured in the lab frame at $t = 0$? What is the laser emission power P measured in the lab frame at $t = 0$? The laser continues to radiate at constant power P_0 (as measured in its rest frame), and recoil from the emitted photons decelerates the ship. Find the mass of the ship when it stops in the lab frame. Find the proper time τ (measured by the clock on the ship) after which the ship comes to a stop in the lab frame.

16. A narrowly collimated laser beam with power P_{in} and wavelength λ is directed at a pair of identical partly reflecting mirrors, each facing each other and having power reflectivity r. These mirrors are a distance L apart. Calculate the power of the transmitted beam.